CS 173, Fall 2014
Examlet 8, Part A

NETID:

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Discussion: **Thursday** **2** **3** **4** **5** **Friday** **9** **10** **11** **12** **1** **2**

(20 points) Suppose that $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is defined by is defined by

$f(1) = 5 \qquad f(2) = -5$

$f(n) = 4f(n-2) - 3f(n-1)$, for all $n \geq 3$

Use induction to prove that $f(n) = 2 \cdot (-4)^{n-1} + 3$

Proof by induction on n.

Base case(s):

Solution: For $n = 1$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^0 + 3 = 2 \cdot 1 + 3 = 5$, which is equal to $f(1)$.

For $n = 2$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^1 + 3 = 2 \cdot (-4) + 3 = -5$, which is equal to $f(2)$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution:

Suppose that $f(n) = 2 \cdot (-4)^{n-1} + 3$, for $n = 1, 2, \ldots, k-1$, for some integer $k \geq 3$

Rest of the inductive step:

Solution:

Using the definition of f and the inductive hypothesis, we get

$f(k) = 4f(k-2) - 3f(k-1) = 4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3)$

Simplifying the algebra,

$$\begin{aligned} 4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3) &= 8 \cdot (-4)^{k-3} + 12 - 6 \cdot (-4)^{k-2} - 9 \\ &= -2 \cdot (-4)^{k-2} - 6 \cdot (-4)^{k-2} + 3 \\ &= -8 \cdot (-4)^{k-2} + 3 = 2 \cdot (-4)^{k-1} + 3 \end{aligned}$$

So $f(k) = 2 \cdot (-4)^{k-1} + 3$, which is what we needed to prove.

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(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by:

$f(1) = 3 \qquad f(2) = 7$

$f(n) = f(n-1) + 2f(n-2)$, for all $n \geq 3$

Use induction to prove that $f(n) \leq 3^n$

Proof by induction on n.

Base case(s):

Solution:

For $n = 1$, $f(n) = 3$ and $3^n = 3$, so $f(n) \leq 3^n$.

For $n = 2$, $f(n) = 7$ and $3^n = 3^2 = 9$, so $f(n) \leq 3^n$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution:

Suppose that $f(n) \leq 3^n$, for $n = 1, 2, \ldots, k-1$, for some integer $k \geq 3$.

Rest of the inductive step:

Solution:

By the inductive hypothesis, we know that $f(k-1) \leq 3^{k-1}$ and $f(k-2) \leq 3^{k-2}$. So, using these two inequalities plus the definition of f, we get:

$f(k) = f(k-1) + 2f(k-2) \leq 3^{k-1} + 2 \cdot 3^{k-2}$

But then

$3^{k-1} + 2 \cdot 3^{k-2} \leq 3^{k-1} + 2 \cdot 3^{k-1} = 3 \cdot 3^{k-1} = 3^k$

So $f(k) \leq 3^k$, which is what we needed to show.

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(20 points) Suppose that $P : \mathbb{N} \to \mathbb{N}$ is defined by

$P(0) = 2 \qquad P(1) = 1$

$P(n) = P(n-1) + 6P(n-2)$, for all $n \geq 2$

Use induction to prove that $P(n) = 3^n + (-2)^n$

Proof by induction on n.

Base case(s):

Solution:

$n = 0$: $P(0) = 2$. Also $3^n + (-2)^n = 3^0 + (-2)^0 = 1 + 1 = 2$. So the claim holds at $n = 0$.

$n = 1$: $P(1) = 1$. Also $3^n + (-2)^n = 3^1 + (-2)^1 = 3 - 2 = 1$. So the claim holds at $n = 1$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution:

Suppose that $P(n) = 3^n + (-2)^n$ for $n = 0, 1, \ldots, k-1$, for some integer $k \geq 2$.

Rest of the inductive step:

Solution:

$$\begin{aligned}
P(k) &= P(k-1) + 6P(k-2) && \text{by the definition of P} \\
&= (3^{k-1} + (-2)^{k-1}) + 6(3^{k-2} + (-2)^{k-2}) && \text{by the inductive hypothesis} \\
&= 3^{k-1} + (-2)^{k-1} + 6 \cdot 3^{k-2} + 6 \cdot (-2)^{k-2} \\
&= 3^{k-1} + (-2)^{k-1} + 2 \cdot 3^{k-1} - 3 \cdot (-2)^{k-1} \\
&= 3 \cdot 3^{k-1} - 2 \cdot (-2)^{k-1} \\
&= 3^k + (-2)^k
\end{aligned}$$

So $P(k) = 3^k + (-2)^k$, which is what we needed to show.

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(20 points) Suppose that $g : \mathbb{N} \to \mathbb{R}$ is defined by

$$g(0) = 0 \qquad g(1) = \tfrac{4}{3}$$

$$g(n) = \tfrac{4}{3}g(n-1) - \tfrac{1}{3}g(n-2), \quad \text{for } n \geq 2$$

Use induction to prove that $g(n) = 2 - \dfrac{2}{3^n}$

Proof by induction on n.

Base case(s):

Solution: $n = 0$: $2 - \frac{2}{3^n} = 2 - \frac{2}{1} = 0 = g(0)$

$n = 1$: $2 - \frac{2}{3^n} = 2 - \frac{2}{3} = \frac{4}{3} = g(1)$

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Solution: Suppose that $g(n) = 2 - \frac{2}{3^n}$, for $n = 0, 1, \cdots, k-1$ for some integer $k \geq 2$.

Inductive Step:

Solution: We need to show that $g(k) = 2 - \dfrac{2}{3^k}$

$$\begin{aligned}
g(k) &= \frac{4}{3}g(k-1) - \frac{1}{3}g(k-2) && \text{[by the def, } k \geq 2\text{]} \\
&= \frac{4}{3}\left(2 - \frac{2}{3^{k-1}}\right) - \frac{1}{3}\left(2 - \frac{2}{3^{k-2}}\right) && \text{[Inductive Hypothesis]} \\
&= \frac{8}{3} - \frac{8}{3^k} - \frac{2}{3} + \frac{2}{3^{k-1}} \\
&= \frac{6}{3} - \frac{8}{3^k} + \frac{6}{3^k} \\
&= 2 - \frac{2}{3^k}.
\end{aligned}$$

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(20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$$f(0) = 1 \qquad f(1) = -5$$

$$f(n) = -7f(n-1) - 10f(n-2), \quad \text{for } n \geq 2$$

Use induction to prove that $f(n) = (-1)^n \cdot 5^n$

Proof by induction on n.

Base case(s):

Solution: $f(0) = 1 = (-1)^0 \cdot 5^0$ and $f(1) = -5 = (-1)^1 \cdot 5^1$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution: Suppose that $f(n) = (-1)^n * 5^n$ for $n = 0, 1, \ldots, k-1$, for some integer $k \geq 2$.

Rest of the inductive step:

Solution: From the inductive hypothesis, we know that $f(k-1) = (-1)^{k-1} * 5^{k-1}$ and $f(k-2) = (-1)^{k-2} * 5^{k-2}$

So then we have

$$\begin{aligned}
f(k) &= -7 \cdot f(k-1) - 10 \cdot f(k-2) \\
&= -7 \cdot (-1)^{k-1} * 5^{k-1} + -10 \cdot (-1)^{k-2} * 5^{k-2} \\
&= 7 \cdot (-1)^k * 5^{k-1} - 10 \cdot (-1)^k * 5^{k-2} \\
&= 7 \cdot (-1)^k * 5^{k-1} - 2 \cdot (-1)^k * 5^{k-1} \\
&= 5 \cdot (-1)^k * 5^{k-1} = 5 \cdot (-1)^k * 5^k
\end{aligned}$$

So $f(k) = 5 \cdot (-1)^k * 5^{k-1} = (-1)^k * 5^k$ which is what we needed to show.

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(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$$f(1) = 0 \qquad f(2) = 12$$

$$f(n) = 4 \cdot f(n-1) - 3 \cdot f(n-2), \quad \text{for } n \geq 3$$

Use induction to prove that $f(n) = 2 \cdot 3^n - 6$

Proof by induction on n.

Base case(s):

Solution: For $n = 1$, $f(1) = 0$ and $2 \cdot 3^n - 6 = 2\dot{3} - 6 = 0$. So the claim is true.

For $n = 2$, $f(2) = 12$ and $2 \cdot 3^n - 6 = 2\dot{3}^2 - 6 = 18 - 6 = 12$. So the claim is true.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution: Suppose that $f(n) = 2 \cdot 3^n - 6$ for $n = 1, 2, \ldots, k-1$ for some positive integer $k \geq 3$.

Rest of the inductive step:

Solution: $f(k) = 4 \cdot f(k-1) - 3 \cdot f(k-2)$ by the definition of f.

So $f(k) = 4 \cdot (2 \cdot 3^{k-1} - 6) - 3 \cdot (2 \cdot 3^{k-2} - 6)$ by the inductive hypothesis.

So $f(k) = 8 \cdot 3^{k-1} - 24 - 6 \cdot 3^{k-2} + 18 = 8 \cdot 3^{k-1} - 2 \cdot 3^{k-1} - 6 = 6 \cdot 3^{k-1} - 6 = 2 \cdot 3^k - 6$

So $f(k) = 2 \cdot 3^k - 6$ which is what we needed to show.