CS 173, Fall 2014
Examlet 5, Part A

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FIRST: | **LAST:**

Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (f(x) - y, 5y + 3)$. Prove that g is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (a, b) be pairs of integers and suppose that $g(x, y) = g(a, b)$.

   By the definition of g, we know that $f(x) - y = f(a) - b$ and $5y + 3 = 5b + 3$. Since $5y + 3 = 5b + 3$, $5y = 5b$, so $y = b$. Substituting this back into the $f(x) - y = f(a) - b$, we find $f(x) - y = f(a) - y$, so $f(x) = f(a)$. Since f is one-to-one, this implies that $x = a$.

   Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in M, there is an element x in C such that $g(x) = y$.

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1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (f(x) + y, y + 3)$. Prove that g is onto.

   Solution: Suppose that (a, b) is a pair of integers.

   Consider $c = a - b + 3$. c is an integer, since a and b are integers. Since f is onto, this means there is an integer x such that $f(x) = c$.

   Now, let $y = b - 3$. We can then calculate:

   $$g(x, y) = (f(x) + y, y + 3) = (c + y, (b - 3) + 3) = ((a - b + 3) + (b - 3), b) = (a, b)$$

   So we've found a point (x, y) such that $g(x, y) = (a, b)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

   Solution: For every elements x and y in C, if $g(x) = g(y)$, then $x = y$

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1. (10 points) Suppose that $g : \mathbb{N} \to \mathbb{N}$ is one-to-one. Let's define the function $f : \mathbb{N}^2 \to \mathbb{N}^2$ by the equation $f(x, y) = (x + g(y), g(x))$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (a, b) be pairs of natural numbers and suppose that $f(x, y) = f(a, b)$.

   By the definition of f, we know that $x + g(y) = a + g(b)$ and $g(x) = g(a)$.

   Since g is one-to-one and $g(x) = g(a)$, $x = a$. Substituting this into $x + g(y) = a + g(b)$, we get $x + g(y) = x + g(b)$, so $g(y) = g(b)$.

   Since g is one-to-one, $g(y) = g(b)$ implies that $y = b$.

   Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in C, there is an element x in M such that $g(x) = y$.

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1. (10 points) Let $g : \mathbb{N} \to \mathbb{N}$ be onto, and let $f : \mathbb{N}^2 \to \mathbb{Z}$ be defined by

$$f(n, m) = (m - 1)g(n)$$

Prove that f is onto.

Solution: Let a be an integer.

Case 1) $a \geq 0$. Since g is onto, we can find a natural number n such that $g(n) = a$. Let $m = 2$. Then $f(n, m) = (2 - 1)g(n) = 1 \cdot a = a$.

Case 2) $a \leq 0$. Then $(-a)$ is a natural number. Since g is onto, we can find a natural number n such that $g(n) = (-a)$. Let $m = 0$. Then $f(n, m) = (0 - 1)g(n) = (-1) \cdot (-a) = a$.

So we've found a point (n, m) such that $g(n, m) = a$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : M \to C$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

Solution: For every elements x and y in M, if $g(x) = g(y)$, then $x = y$