# Midterm Review

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## 1 Grammar

### 1.1

Consider the following grammar, with set of non-terminals $\{S, T, R, X\}$ and set of terminals $\{a, b, \epsilon\}$. $S$ is the start symbol.

$$\begin{aligned}
\texttt{S} &\rightarrow XSX \mid T \\
T &\rightarrow aRb \mid bRa \\
R &\rightarrow XR \mid \epsilon \\
X &\rightarrow a \mid b
\end{aligned}$$

This grammar enforces that any string generated has one of the following forms

$$\texttt{X}\ldots\texttt{XaX}\ldots\texttt{XbX}\ldots\texttt{X}$$

$$\texttt{X}\ldots\texttt{XbX}\ldots\texttt{XaX}\ldots\texttt{X}$$

where the there are equal number of $X$s (possibly 0) on the "oustide" and some arbitrary number of $X$s (possibly 0) in the middle. (Each $X$ becomes a single character $a$ or $b$.) This ensures that any string generated by this grammar is *not a palindrome*.

#### 1.1.1

Draw parse trees for each of the following strings:

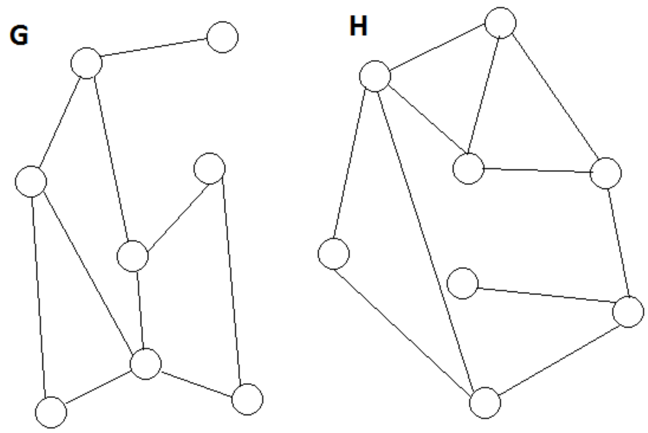
1. ab

2. abb

3. ababbba

# 2 Graphs

## 2.1




### 2.1.1

Are $G$ and $H$ connected?

### 2.1.2

What are the maximum degrees of $G$ and $H$?

### 2.1.3

What are the chromatic numbers of $G$ and $H$? Justify your answer.

### 2.1.4

Are $G$ and $H$ isomorphic? Justify your answer.

## 2.2

If $G$,$H$ are graphs, here are their respective adjacency matrices:

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \ A_H = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

### 2.2.1

Are $G$ and $H$ connected?

### 2.2.2

What are the maximum degrees of $G$ and $H$?

### 2.2.3

What are the chromatic numbers of the $G$ and $H$? Justify your answer.

### 2.2.4

Do $G$ and $H$ contain Eulerian circuits? Justify your answer.

### 2.2.5

Are $G$ and $H$ isomorphic? Justify your answer.

# 3 Induction

## 3.1

Let $\sigma_n$ denote the sum of the digits of $n$ . For example $\sigma_{37} = 10$, $\sigma_{543} = 12$. Prove that $n \equiv \sigma_n \ (mod\ 9)$ using induction. (Hint: What happens if $n$ has a 9 in the singles digit, and we increase $n$ by 1?)

## 3.2

Let the function $f : N \rightarrow Z$ be defined by
$f(0) = 1$
$f(1) = 6$
$\forall n \geq 2, f(n) = 6f(n-1) - 9f(n-2)$
Use strong induction on n to prove that $\forall n \geq 0, f(n) = (1+n)3^n$

## 3.3

### 3.3.1

Prove that $n^0$ is $O(n^1)$.

### 3.3.2

Prove that $n^z$ is $O(n^{z+1})$ for any $z \in \mathbb{N}$.

# 4 Recursion Trees

## 4.1

Find a closed form for
$T(n) = 2T(\frac{n}{3}) + n + 1$
$T(1) = 2$ when $n$ is a power of 3.

## 4.2

Find a closed form for
$T(n) = 2T(n-1) + 3$
$T(1) = 3$

## 4.3

Find a closed form for
$T(n) = T(\lfloor\sqrt{n}\rfloor) + n$
$T(1) = 2$
when $n = 2^{(2^k)}$ for some non-negative integer $k$.

# 5 Big O

## 5.1

Order the following functions in increasing (in terms of Big O) order.

$$n^2,\ n\log_3^2 n,\ \sqrt{n^5+\log_2 n},\ \log_{10} n,\ \log_5 n,\ 2^{\log_3 n},\ (\sqrt{2})^n$$

## 5.2

Find $k \in \mathbb{N}$ such that $n^{\sqrt{2}}\log^2 n$ is $O(n^k)$. Is your bound "tight"?

## 5.3

Prove that $\sqrt{x^4+6x^2+9}$ is $O(x^2)$. Find a $C, k \geq 0$ such that $\sqrt{x^4+6x^2+9} \leq Cx^2$ for all $x \geq k$.

### 5.4

Prove that $n^3 + 7n^2$ is not $O(n^2)$. Given a $C, k \geq 0$, find an $n \geq k$ such that $n^3 + 7n^2 > Cn^2$.