CS 173, Spring 2012, Midterm 2 Solutions

There were two versions of the exam. We've included one or both versions of a question, depending on the extent of the differences between them.

Problem 1: Multiple Choice and True/False (12 points)

Check the box that <u>best</u> characterizes each item. Check only one box per statement. If you change your answer, make sure its easy to tell which box is your final selection.

If a graph has an Euler circuit, then all nodes must have even degree.	**true**	$\boxed{\surd}$	**false**	$\boxed{}$
The number of leaves in a binary tree of height h is at most 2^h.	**true**	$\boxed{\surd}$	**false**	$\boxed{}$
$\displaystyle\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$	**true**	$\boxed{\surd}$	**false**	$\boxed{}$
Two graphs are isomorphic if there is a bijection between their nodes.	**true**	$\boxed{}$	**false**	$\boxed{\surd}$

$g : \mathbb{Z} \to \mathbb{Z}$, $g(x) = 7 - \left\lfloor \frac{x}{3} \right\rfloor$

yes onto yes 1-to-1 $\boxed{}$	yes onto not 1-to-1 $\boxed{\surd}$	not onto yes 1-to-1 $\boxed{}$	not onto not 1-to-1 $\boxed{}$

$f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 3$ if x is even, and $f(x) = x - 22$ if x is odd

yes onto yes 1-to-1 $\boxed{}$	yes onto not 1-to-1 $\boxed{}$	not onto yes 1-to-1 $\boxed{}$	not onto not 1-to-1 $\boxed{\surd}$

The number of leaves in a binary tree of height h is at least 2^h.	**true** ☐	**false** ☑
$\sum_{k=0}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$	**true** ☐	**false** ☑
Any graph with maximum degree d can be colored with d colors.	**true** ☐	**false** ☑
Every bipartite graph is 2-colorable	**true** ☑	**false** ☐

$g : \mathbb{Z} \to \mathbb{R}$, $g(x) = 7 - \left(\frac{x}{3}\right)$

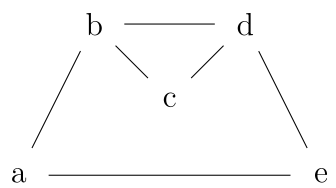
yes onto yes 1-to-1 ☐	yes onto not 1-to-1 ☐	not onto yes 1-to-1 ☑	not onto not 1-to-1 ☐

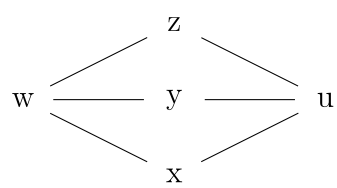
$f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 8$ if x is even, and $f(x) = x - 22$ if x is odd

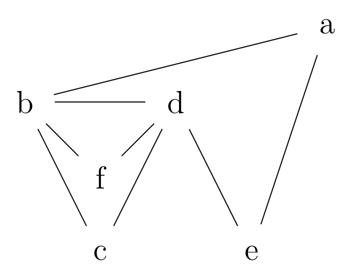
yes onto yes 1-to-1 ☑	yes onto not 1-to-1 ☐	not onto yes 1-to-1 ☐	not onto not 1-to-1 ☐

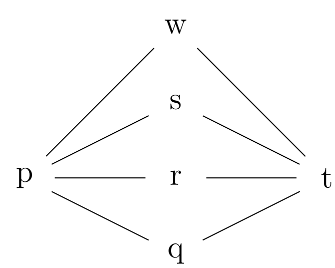
Problem 2: Short Answer (9 points)

(a) (4 points) Are these two graphs isomorphic? Justify your answer.



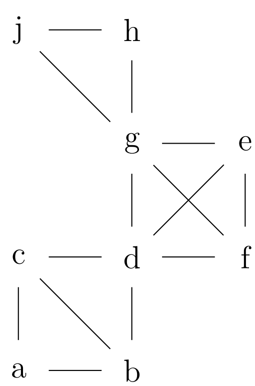


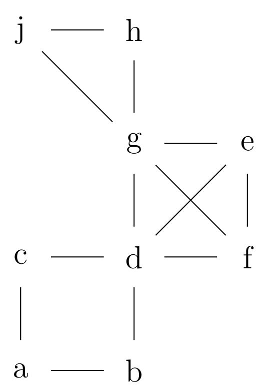




Solution: In both versions, the graphs are not isomorphic. For the top pair [bottom pair]: in the graph on the left the two degree-3 [degree-4] nodes are adjacent, but the two degree-3 [degree-4] nodes are not adjacent in the right graph. Or in the graph on the right none of the degree-2 nodes are adjacent, but two of the degree-2 nodes are adjacent in the left graph. Or there is one C_3 [are two C_3's] in the left graph, but there are no C_3's in the right graph. Or the right graph is bipartite and hence 2-colorable, but the left graph needs at least 3 colors for a coloring. Or there are six [twelve] C_4's in the right graph and only one [two] C_4's in left. Or there is one C_5 [are two C_5's] in the left graph, but there are no C_5's in the right graph.

(b) (5 points) In the following graph, how many different paths are there from c to f on the left [from g to b on the right]? Remember that a path can not repeat vertices. Show your work.





Solution: A path from c to f [from g to b], must pass through d. And it can't go through j and h because then it would have to hit g twice.

On the left: There are three paths from c to d. And then five paths from d to f. Therefore, there are 15 paths from c to f.

On the right: There are five paths from g to d. And then two paths from d to b. Therefore there are 10 paths from g to b.

Problem 3: Recursion Trees (6 points)

Consider the function defined by:

$$f(1) = 13$$

$$f(n) = 5f\left(\frac{n}{6}\right) + n^3, \text{ for } n \geq 2$$

Or, in the second version

$$g(1) = 11$$

$$g(n) = 3g\left(\frac{n}{7}\right) + n^2, \text{ for } n \geq 2$$

(a) What is an appropriate domain for this function, i.e. for which input values does this definition provide a well-defined output value? (For parts (b)-(d), you should assume all inputs are from this set.)

Solution: We need to ensure that the input to f [g] is always an integer. So each input must be a power of 6 [7]. So the domain is the natural number powers of 6 [powers of 7].

(b) The height h of the recursion tree for this function is:

Solution: The input at level h is $\frac{n}{6^h}$ [$\frac{n}{7^h}$] this reaches the base part 1 of the definition precisely when $h = \log_6 n$ [$\log_7 n$]

(c) The sum of all the values at level k (where $k < h$) of the tree is:

Solution: There are 5^k [3^k] nodes at level k. The input is $\frac{n}{6^k}$ [$\frac{n}{7^k}$]. So the value in each node is $(\frac{n}{6^k})^3$ [$(\frac{n}{7^k})^2$]. Therefore, the sum of all the values at level k is $5^k(\frac{n}{6^k})^3$ [$3^k(\frac{n}{7^k})^2$].

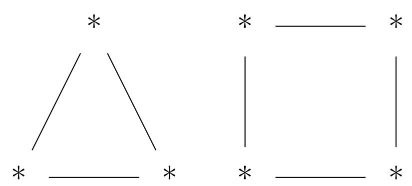
(d) Give the sum of all the leaf values. You do not need to "simplify" your formula.

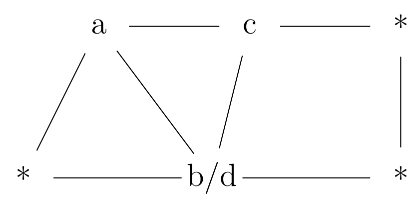
Solution: There are $5^{\log_6 n}$ [$3^{\log_7 n}$] leaves, each containing the value 13 [11]. So the sum of all the leaf values is $13 \cdot 5^{\log_6 n}$ [$11 \cdot 3^{\log_7 n}$].

Problem 4: Graphs (6 points)

Recall that if G is a graph, then $\chi(G)$ is its chromatic number. Suppose that G and H are each connected graphs but H is not connected to G. Suppose also that G and H each have at least two nodes and at least one edge. Dr. Evil merges G and H into a single graph T as follows. He picks two adjacent vertices a and b from G, and also two adjacent vertices c and d from H. He adds an edge connecting a and c, and merges b and d into a single vertex.

For example, if G and H are as shown on the left, then T might look as shown on the right.





Note: Solution for one version is shown. The other version differs only in what points/graphs have which names. As long as you gave the key points about the special case, it doesn't matter whether you put them in part 1, part 2, or partly in both places.

1. How is $\chi(T)$ related to $\chi(G)$ and $\chi(H)$? (Be sure to address any special cases.)

   Solution: $\chi(T) = \max(\chi(G), \chi(H), 3)$

2. Justify your answer:

   Solution: Notice that the output graph contains a triangle, so it definitely requires at least three colors.

   Without loss of generality, suppose that $k = \chi(G) \geq \chi(H)$. Then $\chi(T)$ must be at least k because G is a subgraph of T. Also notice that k is at least 2 because the two input graphs each contain an edge.

   First, suppose k is at least 3. To color T with k colors, first color the part of T corresponding to G. We have a coloring of H that uses $\leq k$ colors, but the color choices might not be compatible with how we've started coloring T. If the two merged nodes b and d have different colors, swap the names of two colors to make them same. If a and c have the same color, swap the color of c with some third color, remembering that k is at least 3. Adjust the rest of the coloring for H to use these same choices of color names.

   Special case: if $k = 2$, then we carry out the same procedure. However, we won't have any third color available to fix the color of c, so we'll have to allocate an extra color.

Problem 5: Tree Induction (7 points)

Changes for the second version are shown in square brackets.
Use strong induction to prove that a full 5-ary [4-ary] tree of height h has at least $4h + 1$ [$3h + 1$] leaves.
(Recall that a m-ary tree is full if each node has either zero or m children.)

Solution:

Base Case(s):
When the height $h = 0$ then the tree is one single node, which is itself a leaf, and yes it has at least 1= 4(0)+1 = 4h+1 [1=3(0)+1 = 3h+1] leaves.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
Assume for any full 5-ary [4-ary] tree T of height k for $k = 0, 1, \cdots, h-1$ where $h \geq 1$, that T has at least $4k + 1$ [$3k + 1$] leaves.

Inductive Step:
Consider a full 5-ary (4-ary) tree T of height $h > 0$. Then the root is not a leaf, and because T is full, it must contain exactly 5 [4] subtrees, T_1, ..., T_5 [...T_4]. At least one of these trees must have height $h - 1$, and the others must have height at least 0.
By the inductive hypothesis, the subtree with height $h - 1$ has at least $4(h - 1) + 1$ [$3(h - 1) + 1$] leaves, and the other 4 [3] subtrees have at least one leaf each.
Hence, the total number of leaves is at least:
$(4(h - 1) + 1) + 4 \cdot 1 = 4h - 4 + 5 = 4h + 1$
[$(3(h - 1) + 1) + 3 \cdot 1 = 3h - 3 + 4 = 3h + 1$].
This completes the induction proof.

Problem 6: Induction (10 points)

Let the function $f : \mathbb{N} \to \mathbb{R}$ be defined by

$$f(0) = \tfrac{2}{3}$$

$$f(1) = \tfrac{8}{9}$$

$$f(n) = \tfrac{4}{3}f(n-1) - \tfrac{1}{3}f(n-2), \quad \text{for } n \geq 2$$

Use strong induction on n to prove that $f(n) = 1 - \frac{1}{3^{n+1}}$ for any natural number n.

Solution:

Base Case(s):
$n = 0$: $1 - \frac{1}{3^{n+1}} = 1 - \frac{1}{3} = \frac{2}{3} = f(0)$
$n = 1$: $1 - \frac{1}{3^{n+1}} = 1 - \frac{1}{9} = \frac{8}{9} = f(1)$

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $f(n) = 1 - \frac{1}{3^{n+1}}$, for $n = 0, 1, \cdots, k-1$ where $k \geq 2$.

Inductive Step:
We need to show that $f(k) = 1 - \dfrac{1}{3^{k+1}}$

$$\begin{aligned}
f(k) &= \frac{4}{3}f(k-1) - \frac{1}{3}f(k-2) && \text{[by the def, } k \geq 2\text{]} \\
&= \frac{4}{3}\left(1 - \frac{1}{3^k}\right) - \frac{1}{3}\left(1 - \frac{1}{3^{k-1}}\right) && \text{[Inductive Hypothesis]} \\
&= \frac{4}{3} - \frac{4}{3^{k+1}} - \frac{1}{3} + \frac{1}{3^k} \\
&= 1 - \frac{4}{3^{k+1}} + \frac{3}{3^{k+1}} \\
&= 1 - \frac{1}{3^{k+1}}.
\end{aligned}$$

Problem 6: Induction (10 points)

Let the function $g : \mathbb{N} \to \mathbb{R}$ be defined by

$$g(0) = 0$$

$$g(1) = \tfrac{4}{3}$$

$$g(n) = \tfrac{4}{3}g(n-1) - \tfrac{1}{3}g(n-2), \quad \text{for } n \geq 2$$

Use strong induction on n to prove that $g(n) = 2 - \frac{2}{3^n}$ for any natural number n.

Base Case(s):
$n = 0$: $2 - \frac{2}{3^n} = 2 - \frac{2}{1} = 0 = g(0)$
$n = 1$: $2 - \frac{2}{3^n} = 2 - \frac{2}{3} = \frac{4}{3} = g(1)$

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $g(n) = 2 - \frac{2}{3^n}$, for $n = 0, 1, \cdots, k-1$ where $k \geq 2$.

Inductive Step:
We need to show that $g(k) = 2 - \dfrac{2}{3^k}$

$$\begin{aligned}
g(k) &= \frac{4}{3}g(k-1) - \frac{1}{3}g(k-2) && \text{[by the def, } k \geq 2\text{]} \\
&= \frac{4}{3}\left(2 - \frac{2}{3^{k-1}}\right) - \frac{1}{3}\left(2 - \frac{2}{3^{k-2}}\right) && \text{[Inductive Hypothesis]} \\
&= \frac{8}{3} - \frac{8}{3^k} - \frac{2}{3} + \frac{2}{3^{k-1}} \\
&= \frac{6}{3} - \frac{8}{3^k} + \frac{6}{3^k} \\
&= 2 - \frac{2}{3^k}.
\end{aligned}$$