

CS 173, Final Exam

7 May 2012

Fill in your name, netid (eg: alincoln16), and circle your lecture and discussion section below. Also write your name or netid on the last page (which sometimes gets pulled off).

Printed Name:

NetID:

Circle your Assigned Lecture:

Lec	time	Professor
AL1	11:00	Lenny Pitt
AL2	12:30	Margaret Fleck
Proficiency	write your UIN below	

Circle your Discussion Section:

Sec	time	TA's name
ADA	Thr	2 Andrew
ADB	Thr	3 Andrew
ADC	Thr	4 Kevin
ADD	Thr	5 Kevin/Yaonan
ADE	Thr	6 Ryan
ADF	Thr	7 Ryan/Tom
ADG	Fri	9 Leila/Sarah
ADH	Fri	10 Leila
ADI	Fri	11 Parth
ADJ	Fri	12 Lance
ADK	Fri	1 Lance
ADL	Fri	2 Fatimeh
ADM	Fri	3 Fatimeh/Maya

Scores:

1		/12
2		/15
3		/12
4		/12
5		/10
6		/13
7		/13
8		/13

Totals:

1-4		/51
5-8		/49
1-8		/100

INSTRUCTIONS: please read carefully

- There are 8 problems, each on a single page. Make sure you have a complete exam.
- The point value of each problem is indicated next to the problem, and on the cover page table. It is wise to skim all problems and point values first, to best plan your time.
- Points may be deducted for solutions which are correct, but hard to read, hard to understand, poorly explained, or excessively complicated.
- For proofs, be sure to use good mathematical style, with the steps in logical order.
- Brief explanations and/or showing work, even when not explicitly asked for, may increase partial credit for buggy answers. Partial credit for multiple-choice questions is very rare.
- It is not necessary to simplify or calculate out complex constant expressions such as $(0.7)^3(0.3)^5$, $\frac{0.15}{3.75}$, 3^{17} , and $7!$, unless it is explicitly indicated to completely simplify.
- This is a closed book exam.
Turn off your cell phone now.
No notes or electronic devices of any kind are allowed.
These should be secured in your bag and out of reach during the exam.
- Do all work in the space provided, using the backs of sheets if necessary.
If your work is on the backside then you must clearly indicate so on the problem.
Ask a proctor if you need more paper.
- Please bring any apparent bugs or ambiguity to the attention of the proctors.
- We expect most people to finish the exam in 2 hours, but you may take up to the full 3 hours.
- When you are finished, show your ID to the proctors and put your exam in the appropriate marked pile at the front.
- Because of conflict exams (including last-minute emergencies) do not discuss the contents of the exam with other students until the end of the final exam period, i.e. Saturday May 12th.

Problem 1: Multiple choice (12 points)

Check the box that best characterizes each item.

$\mathbb{P}(\emptyset) =$

\emptyset	<input type="checkbox"/>	$\{\emptyset\}$	<input type="checkbox"/>	$\{\{\emptyset\}\}$	<input type="checkbox"/>
$\{\emptyset, \{\emptyset\}\}$	<input type="checkbox"/>	$\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$	<input type="checkbox"/>	undefined	<input type="checkbox"/>

Dividing a problem of size n into k sub-problems, each of size n/m , has the best big- Θ running time when

$k < m$	<input type="checkbox"/>	$k = m$	<input type="checkbox"/>
$k > m$	<input type="checkbox"/>	$km = 1$	<input type="checkbox"/>

The running time of the Towers of Hanoi solver is recursively defined by $T(1) = 1$ and $T(n) =$

$2T(n-1) + c$	<input type="checkbox"/>	$2T(n-1) + cn$	<input type="checkbox"/>
$2T(n/2) + c$	<input type="checkbox"/>	$2T(n/2) + cn$	<input type="checkbox"/>

If A is a countably infinite set, then $\mathbb{P}(A)$ is uncountable.

true	<input type="checkbox"/>	depends on A	<input type="checkbox"/>
false	<input type="checkbox"/>	depends on \mathbb{P}	<input type="checkbox"/>

In 1940, Rutenbar and Flores published a simple inductive proof that any planar graph is 4-colorable.

true	<input type="checkbox"/>	false	<input type="checkbox"/>
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The cube graph Q_3 is planar

true	<input type="checkbox"/>	it depends on	
false	<input type="checkbox"/>	how you draw Q_3	<input type="checkbox"/>

Problem 2: Short answer (15 points)

- (a) (5 points) Suppose that f and g are functions from \mathbb{R} to \mathbb{R} . Using precise mathematical language, define what it means for f to be $O(g)$.
- (b) (5 points) Suppose that a planar graph has v nodes, e edges, f faces, and k connected components. State a (simple) generalization of Euler's formula that relates v , e , f , and k . Briefly justify your answer.
- (c) (5 points) If G is a graph, then \overline{G} , the complement of G , is the graph with the same vertex set as G , and such that for each pair of distinct vertices u and v , there is an edge from u to v in \overline{G} if and only if there is not such an edge in G .
- Suppose that G is a tree with n vertices. How many edges does \overline{G} have? Briefly but clearly justify your answer.

Problem 3: Short answer (12 points)

- (a) (6 points) Remember that a phone lattice is a type of state diagram, i.e. a directed graph where each node represents one or more states of a system. A phone lattice has exactly one letter on each edge. Each path from the start node to a final/end node represents a word. Draw a phone lattice representing exactly the following set of words, using a single start node and no more than 8 nodes total.

sip, sap, clip, clap, aw, aaw, aaaw, aaaaw, [one or more *a*'s followed by one *w*]

- (b) (6 points) CMU's new robotic hummingbird Merrill travels in 3D. Each command makes it move one foot in a specified cardinal direction e.g. up/down, north/south, or east/west, but not diagonally. How many different sequences of 30 commands will get Merrill from position (1, 10, 3) to position (10, 6, 20)? Briefly explain your answer and/or show work.

Problem 4: Short answer (12 points)

(a) (5 points) Let's say that two graphs are distinct if and only if they are not isomorphic. Is the set of distinct (finite) planar graphs countable or uncountable? Briefly justify your answer.

(b) Let $A = \{2, 5, 7, 8, 13, 21\}$. Define $p : A \rightarrow \mathbb{P}(A)$ by $p(n) = \{s \in A \mid \gcd(s, n) \neq 1\}$.

(2 points) Give the value of $p(7)$.

(2 points) Let $M = \{p(s) \mid s \in A\}$. Evaluate/List out the elements of M .

$M = \left\{ \right.$

(3 points) Is M a partition of A ? Justify your answer by explaining why M satisfies, or doesn't satisfy, each of the three defining properties of a partition.

Problem 5: Short answer (10 points)

- (a) (5 points) Give the negation of the following sentence. Your answer should be in mathematical English (not shorthand), with all negations (e.g. “not”) on individual predicates.

There is a plant p , such that for every bug b , if b pollinates p then b has four wings and b is nocturnal.

- (b) (5 points) Put the following 7 functions of n in increasing order according to big- O in the boxes below. That is, write f to the left of g if and only if f is $O(g)$.

$2^n + 3^n$, n^n , $100 \log n$, n^2 , $3n \log(n^3)$, $7n! + 2$, $173n - 173$

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Problem 6: Algorithm analysis (13 points)

Here is the pseudocode for a function `zsort` that claims to sort its input array. The function `swap(a_x, a_y)` exchanges the values in positions x and y .

```
01  zsort( $a_1, \dots, a_n$ ) : array of  $n$  positive integers
02      if ( $n = 2$  and  $a_1 > a_2$ )
03          swap( $a_1, a_2$ )
04      else if ( $n > 2$ )
05           $p = \lfloor \frac{n}{3} \rfloor + 1$ 
06           $q = \lceil \frac{2n}{3} \rceil$ 
07          zsort( $a_1, \dots, a_q$ )
08          zsort( $a_p, \dots, a_n$ )
09          zsort( $a_1, \dots, a_q$ )
```

For simplicity: assume that everything works for non-integer values of n , i.e. do not use the floor and ceiling functions in your answers.

- (a) (4 points) Suppose that $T(n)$ is the running time of `zsort` on an input array of length n . So $T(1) = T(2) = c$ for some constant c . Give a recursive definition of $T(n)$. Assume that the time to set up the recursive calls in steps 7 through 9 is constant.

- (b) (3 points) What is the height of the recursion tree for $T(n)$?

- (c) (4 points) How many leaves are in the recursion tree for $T(n)$? Show your work and use the change of base formula $\log_a x = \log_b x \cdot \log_a b$ to simplify your answer into a more convenient form.

- (d) (2 points) Is the running time of `zsort` $O(n \log n)$? Briefly justify your answer.

Problem 7: Induction (13 points)

Suppose that $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ is defined by

$$\begin{aligned} f(1) &= 1 \\ f(2) &= 5 \\ f(n) &= 5f(n-1) - 6f(n-2), \quad \forall n > 2 \end{aligned}$$

Use induction to prove that $f(n) = 3^n - 2^n$ for all $n \in \mathbb{Z}^+$.

Base case(s): [Be careful about logical order of assertions.]

Inductive Hypothesis: [Give specific details. Do not refer to “*the claim*”.]

Inductive Step:

Write your netID, in case this page gets pulled off:

Problem 8: Tree induction (13 points)

A parity tree is a full binary tree in which each node is colored orange or blue, such that:

- If v is a leaf node, then v is colored orange.
- If v has two children of the same color, then v is colored blue.
- If v has two children of different colors, then v is colored orange.

Prove by induction that every parity tree has the *parity property*: if the root is colored orange, then it has an odd number of leaves; and if the root is colored blue, then it has an even number of leaves.

The induction variable is:

Base case(s):

Inductive Hypothesis: [Be specific. You may use the term “*parity property*”.]

Inductive Step: [You may use basic facts about odd and even, e.g. the sum of two odd numbers is even.]