CS 173 Fall 2012
Review problems for the second midterm

1. **Set Inclusion Proofs**

   Let $f : X \to Y$ be any function, and let A and B be subsets of X. For any subset S of X define its image $f(S)$ by $f(S) = \{f(s) \in Y \mid s \in S\}$.

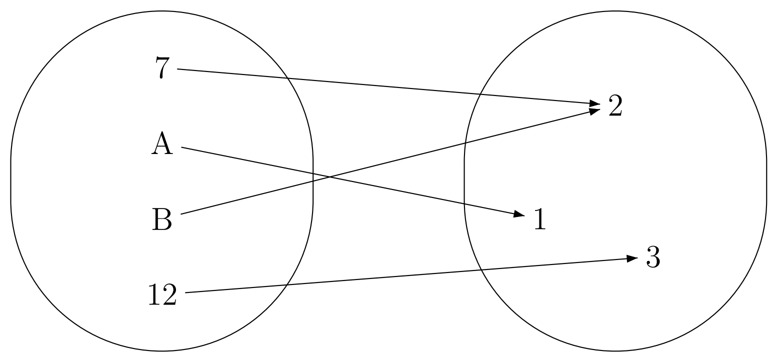
   (a) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$. You **must** use the technique of choosing an element from the smaller set and showing that it is also a member of the larger set.

   (b) Prove that it's not necessarily the case that $f(A) \cap f(B) \subseteq f(A \cap B)$ by giving specific **finite** sets and a specific function for which this inclusion does not hold.

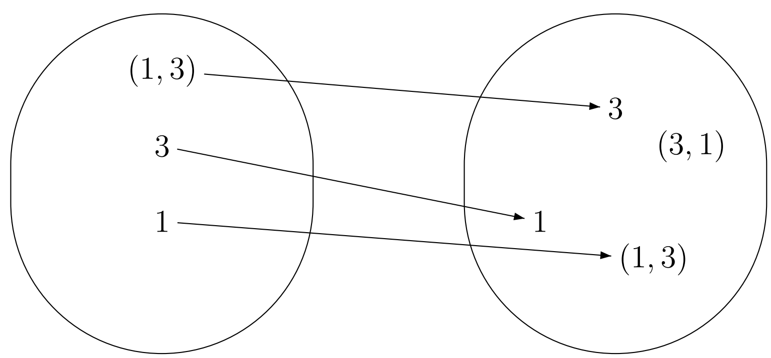
2. **Functions**

   For each of the following functions determine if it is onto or not onto. Briefly, but clearly, justify your answers. (A full formal proof is not required.)

   (a) The function f given by the following diagram where the left bubble represents the domain and the right the codomain:



   (b) The function g given by the following diagram:



   (c) $h : \mathbb{Z} \to \mathbb{Z}$ by $h(x) = 3\lceil \frac{x}{3} \rceil$

   (d) $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $k(x, y) = x$

3. **One-to-one**

   Which of these functions are one-to-one? Briefly justify your answers.

   (a) $h : [0,1] \to \mathbb{R}^2$ such that $h(\lambda) = \lambda(2,2) + (1-\lambda)(1,3)$ where you use the following formula to multiply a real number a by a 2D point (x,y):

   $$a(x,y) = (ax, ay)$$

   (b) $f : \mathbb{N}^2 \to \mathbb{N}$ such that $f(x,y) = 4^x 3^y$

   (c) $k : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ such that $k(x,y) = (1-x^2)\left\lfloor \frac{y}{3} \right\rfloor$

   (d) $g : \mathbb{F} \to \mathbb{R}$ such that $g(a + b\epsilon) = \sqrt{a^2 + b^2}$ where $\mathbb{F}$ is the set of "funny numbers" that contains all numbers of the form $x + y\epsilon$, where $\epsilon > 0$ and $\epsilon^2 = 0$

4. **Nested Quantifiers**

   Prove or disprove the statements in (a), (b), and (d). **Hint**: these proofs/disproofs are meant to be very brief.

   (a) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ \text{GCD}(x,y) = 1$

   (b) $\forall x \in \mathbb{Z}^+,\ \exists y \in \mathbb{Z}^+,\ x = y^2$

   (c) Suppose that f is a function from $\mathbb{Z}_6$ to $\mathbb{Z}_8$, and $\exists c \in \mathbb{Z}_8,\ \forall x \in \mathbb{Z}_6,\ f(x) = c$. Give a one sentence description of the function f.

   (d) $\exists f : \mathbb{Z}_6 \to \mathbb{Z}_8,\ \exists c \in \mathbb{Z}_8,\ \forall x \in \mathbb{Z}_6,\ f(x) = c$

Trees

Check the box that best characterizes each item.

$\sum_{k=0}^{n-1} 2^k$ — $2^n - 2$: ☐ $2^n - 1$: ☐ $2^{n-1} - 1$: ☐ $2^{n+1} - 1$: ☐

The level of the root node in a tree of height h. — 0: ☐ 1: ☐ $h-1$: ☐ h: ☐ $h+1$: ☐

How often is the root node of a tree an internal node? — **never**: ☐ **sometimes**: ☐ **always**: ☐

Short answer

(a) Suppose that $g : A \to B$ and $f : B \to C$. Prof. Snape claims that if $f \circ g$ is onto, then g is onto. Disprove this claim using a concrete counter-example in which A, B, and C are all small finite sets.

(b) Suppose that A, B and C are sets. Recall the definition of $X \subseteq Y$: for every p, if $p \in X$, then $p \in Y$. Prove that if $A \subseteq B$ then $A \cap C \subseteq B \cap C$. Briefly justify the key steps in your proof.

(c) Suppose that $g : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define the function $f : \mathbb{Z} \to \mathbb{Z}^2$ by $f(x) = (x^2, g(x))$. Prove that f is one-to-one.

(d) How many different 6-letter strings can I make out of the letters in the word "illini"?

(e) Define the function f as follows:

- $f(1) = 1$
- $f(2) = 5$
- $f(n+1) = 5f(n) - 6f(n-1)$

Suppose we're proving that $f(n) = 3^n - 2^n$ for every positive integer n. State the inductive hypothesis and the conclusion of the inductive step.

Induction

Let the function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 1$

$f(1) = 6$

$\forall n \geq 2,\ f(n) = 6f(n-1) - 9f(n-2)$

Use strong induction on n to prove that $\forall n \geq 0,\ f(n) = (1+n)3^n$.

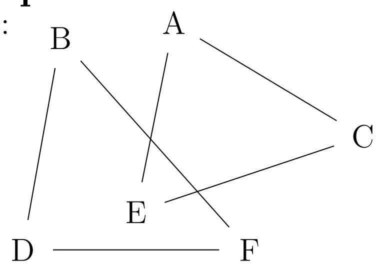
Base case(s):

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

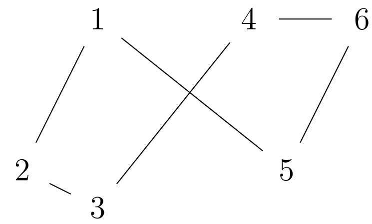
Rest of the inductive step:

Graphs

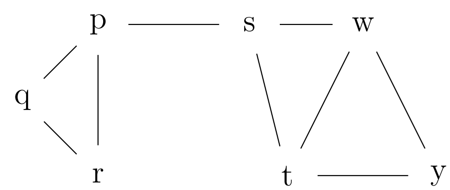
G1:



G2:



G3:



(a) How many connected components does each graph have?

(b) Are graphs G1 and G2 (above) isomorphic? Briefly justify your answer.

(c) What is the diameter of G3?

(d) Does G3 contain an Euler circuit? Why or why not?

(e) Does G2 and/or G3 contain a cut edge? If so, identify which edge(s) are cut edges.