CS 173, Fall 2012 Midterm 2 Review Solutions

1. **Set Inclusion Proofs**

   Let $f : X \to Y$ be any function, and let A and B be subsets of X. For any subset S of X define its image $f(S)$ by $f(S) = \{f(s) \in Y \mid s \in S\}$.

   (a) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$. You **must** use the technique of choosing an element from the smaller set and showing that it is also a member of the larger set.

   Solution: Note that $f(A \cap B)$ and $f(A) \cap f(B)$ are *sets*. Suppose y is an arbitrary element of $f(A \cap B)$. By the definition of the image of a set, there is an element $x \in A \cap B$ such that $f(x) = y$. Since $x \in A \cap B$, we know $x \in A$ and $x \in B$. Thus, $y \in f(A)$ and $y \in f(B)$. So $y \in f(A) \cap f(B)$. Since y was arbitrary, we conclude that $f(A \cap B) \subseteq f(A) \cap f(B)$.

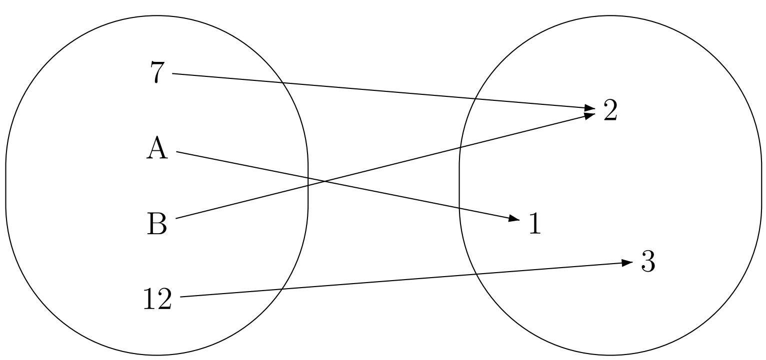
   (b) Prove that it's not necessarily the case that $f(A) \cap f(B) \subseteq f(A \cap B)$ by giving specific **finite** sets and a specific function for which this inclusion does not hold.

   Solution: Let $X = \{a, b\}$ and $Y = \{c\}$. Define $f : X \to Y$ by $f(x) = c$ for all $x \in X$. Choose $A = \{a\}$, and $B = \{b\}$. Then $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$. However, $f(A) = Y = f(B)$, so $f(A) \cap f(B) = Y$.

2. **Functions**

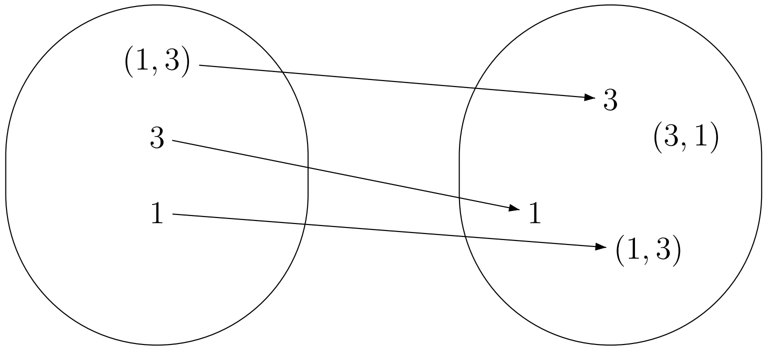
   For each of the following functions determine if it is onto or not onto. Briefly, but clearly, justify your answers. (A full formal proof is not required.)

   (a) The function f given by the following diagram where the left bubble represents the domain and the right the codomain:



   Solution: The function f is onto because every output has at least one corresponding input that the function maps to it.

(b) The function g given by the following diagram:



Solution: The function g is not onto because the codomain element $(3, 1)$ has no corresponding input that maps to it.

(c) $h : \mathbb{Z} \to \mathbb{Z}$ such that $h(x) = 3\lceil \frac{x}{3} \rceil$

Solution: The function h is not onto because 1 is not in the image of the function. If it were, then $1 = 3\lceil \frac{x}{3} \rceil$ which is impossible because $\lceil \frac{x}{3} \rceil$ is an integer.

(d) $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $k(x, y) = x$

Solution: The function k is onto. Pick any codomain element $x \in \mathbb{R}$. Consider $(x, 0) \in \mathbb{R} \times \mathbb{R}$. Notice that $k(x, 0) = x$, so x has a pre-image.

3. **One-to-one**

Which of these functions are one-to-one? Briefly justify your answers.

(a) $h : [0, 1] \to \mathbb{R}^2$ such that $h(\lambda) = \lambda(2, 2) + (1 - \lambda)(1, 3)$ where you use the following formula to multiply a real number a by a 2D point (x, y):

$$a(x, y) = (ax, ay)$$

Solution

h is one-to-one. In $\mathbb{R}^2$, h describes the strictly increasing line segment between the points (2,2) and (1,3).

(b) $f : \mathbb{N}^2 \to \mathbb{N}$ such that $f(x, y) = 4^x 3^y$

Solution

f is one-to-one. The image of f is the set of positive integers that have only 2 and 3 as prime factors and the prime factorization of any integer is unique.

(c) $k : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ such that $k(x, y) = (1 - x^2) \left\lfloor \frac{y}{3} \right\rfloor$

Solution

k is not one-to-one. (0,0) and (1,0) are both mapped to 0.

(d) $g : \mathbb{F} \to \mathbb{R}$ such that $g(a + b\epsilon) = \sqrt{a^2 + b^2}$ where $\mathbb{F}$ is the set of "funny numbers" that contains all numbers of the form $x + y\epsilon$, where $\epsilon > 0$ and $\epsilon^2 = 0$

Solution

g is not one-to-one. -1 (i.e. $-1 + 0\epsilon$) and 1 are both mapped to 1.

4. **Nested Quantifiers**

Prove or disprove the statements in (a), (b), and (d). **Hint**: these proofs/disproofs are meant to be very brief.

(a) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ \text{GCD}(x, y) = 1$

Solution: True. Let $x = 1$. Note that $\text{GCD}(1, y) = 1$ for any choice of y since 1 divides all natural numbers (including 0).

(b) $\forall x \in \mathbb{Z}^+,\ \exists y \in \mathbb{Z}^+,\ x = y^2$

Solution: This says that all positive integers are perfect squares, which is false. Choose $x = 2$. If there were an integer y such that $2 = y^2$, then, $y = \sqrt{3}$ must be an integer, which is absurd.

(c) Suppose that f is a function from $\mathbb{Z}_6$ to $\mathbb{Z}_8$, and $\exists c \in \mathbb{Z}_8,\ \forall x \in \mathbb{Z}_6,\ f(x) = c$. Give a one sentence description of the function f.

Solution: The function f sends all inputs to a single output $c \in \mathbb{Z}_8$, i.e., it is a constant function.

(d) $\exists f : \mathbb{Z}_6 \to \mathbb{Z}_8,\ \exists c \in \mathbb{Z}_8,\ \forall x \in \mathbb{Z}_6,\ f(x) = c$

Solution: True. Let $c = [0]$ and simply take f to be the constant function which sends all inputs $x \in \mathbb{Z}_6$ to $[0] \in \mathbb{Z}_8$, i.e., $f(x) = [0]$ for all $x \in \mathbb{Z}_6$.

Trees

Check the box that best characterizes each item.

$\sum_{k=0}^{n-1} 2^k$

- $2^n - 2$: [ ]
- $2^n - 1$: [√]
- $2^{n-1} - 1$: [ ]
- $2^{n+1} - 1$: [ ]

The level of the root node in a tree of height h.

- 0: [√]
- 1: [ ]
- $h - 1$: [ ]
- h: [ ]
- $h + 1$: [ ]

How often is the root node of a tree an internal node?

- **never**: [ ]
- **sometimes**: [√]
- **always**: [ ]

Short answer

(a) Suppose that $g : A \to B$ and $f : B \to C$. Prof. Snape claims that if $f \circ g$ is onto, then g is onto. Disprove this claim using a concrete counter-example in which A, B, and C are all small finite sets.

Solution: Suppose that $A = \{1, 2\}$, $B = \{3, 4, 5\}$, and $C = \{\text{red}, \text{blue}\}$. Define g by $g(1) = 3$ and $g(2) = 5$. Define f by $f(3) = \text{red}$, $f(4) = \text{red}$, and $f(5) = \text{blue}$.

Then $(f \circ g)(1) = \text{red}$ and $(f \circ g)(2) = \text{blue}$. So $f \circ g$ is onto because every element of C has a pre-image. However, g isn't onto because no element of A maps onto 4.

(b) Suppose that A, B and C are sets. Recall the definition of $X \subseteq Y$: for every p, if $p \in X$, then $p \in Y$. Prove that if $A \subseteq B$ then $A \cap C \subseteq B \cap C$. Briefly justify the key steps in your proof.

Solution: Suppose that $p \in A \cap C$. Then $p \in A$ and $p \in C$, by the definition of intersection. Since $p \in A$ and $A \subseteq B$, $p \in B$ (definition of subset). So $p \in B$ and $p \in C$, which implies that $p \in B \cap C$ (definition of intersection).

(c) Suppose that $g : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define the function $f : \mathbb{Z} \to \mathbb{Z}^2$ by $f(x) = (x^2, g(x))$. Prove that f is one-to-one.

Solution: Let x and y be integers. Suppose that $f(x) = f(y)$. By the definition of f, this means that $(x^2, g(x)) = (y^2, g(y))$. So then $x^2 = y^2$ and $g(x) = g(y)$. Since $g(x) = g(y)$ and g is one-to-one, $x = y$.

So we have that $f(x) = f(y)$ implies $x = y$. This means that f is one-to-one.

(d) How many different 6-letter strings can I make out of the letters in the word "illini"?

Solution: We calculate the number of permuations of 6 letters (6!) and divide out by the double-counting of the possibilities for l (2!) and for i (3!). This gives us $\frac{6!}{2!3!} = 5 \cdot 4 \cdot 3 = 60$ possible strings.

(e) Define the function f as follows:

- $f(1) = 1$
- $f(2) = 5$
- $f(n+1) = 5f(n) - 6f(n-1)$

Suppose we're proving that $f(n) = 3^n - 2^n$ for every positive integer n. State the inductive hypothesis and the conclusion of the inductive step.

Solution: Inductive hypothesis: suppose that $f(n) = 3^n - 2^n$ for $n = 1, 2, \ldots k$, for some integer k.

Conclusion of the inductive step: $f(k+1) = 3^{k+1} - 2^{k+1}$.

Note 1: a strong hypothesis is required because the formula reaches back two integers.

Note 2: the variable k in the conclusion matches the upper bound in the hypothesis. A common mistake is to have it match the variable in the hypothesis equation (n). We're assuming that the equation holds for all values up through k, so we need to prove it holds for $k+1$.

Induction

Let the function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 1$

$f(1) = 6$

$\forall n \geq 2,\ f(n) = 6f(n-1) - 9f(n-2)$
Use strong induction on n to prove that $\forall n \geq 0,\ f(n) = (1+n)3^n$.

Base case(s):

Solution: $f(0) = 1 = (1+0)3^0$ and $f(1) = 6 = (1+1)3^1$. We need to check two base cases because the inductive step will reach back two integers.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution: Suppose that $f(n) = (1+n)3^n$ for $n = 0, 1, \ldots, k$, for some $k \geq 1$.

Rest of the inductive step:

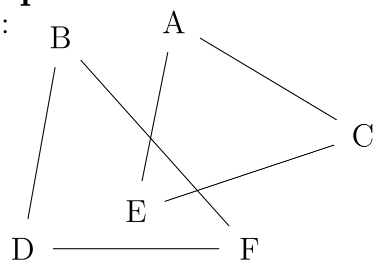
Solution: $f(k+1) = 6f(k) - 9f(k-1)$ by the definition of f. By the inductive hypothesis, we know that $f(k) = (1+k)3^k$ and $f(k-1) = k3^{k-1}$. So by substituting, we get

$$\begin{aligned}
f(k+1) &= 6(1+k)3^k - 9k3^{k-1} \\
&= 2(1+k)3^{k+1} - k3^{k+1} \\
&= 2 \cdot 3^{k+1} + 2 \cdot k3^{k+1} - k3^{k+1} \\
&= 2 \cdot 3^{k+1} + k3^{k+1} \\
&= (k+2)3^{k+1}
\end{aligned}$$

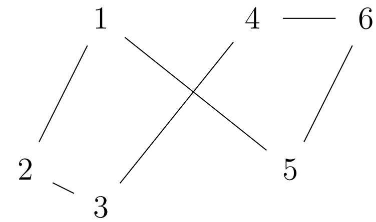
So $f(k+1) = (k+2)3^{k+1}$, which is what we needed to show.

Graphs

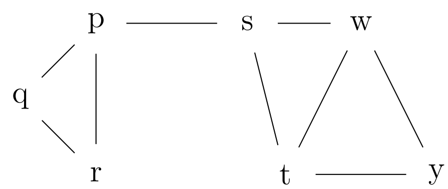
G1:



G2:



G3:



(a) How many connected components does each graph have?

Solution: G1 has two connected components. G2 and G3 each have one connected component.

(b) Are graphs G1 and G2 (above) isomorphic? Briefly justify your answer.

Solution: No. G2 is connected, but G1 isn't connected. Also, G2 contains a cycle with 6 vertices, and G1 doesn't.

(c) What is the diameter of G3?

Solution: 4. (It's the number of edges on a shortest path between the two vertices furthest apart. In this case, y and either q or r.)

(d) Does G3 contain an Euler circuit? Why or why not?

Solution: No, it can't contain an Euler circuit because some of the vertices (e.g. p) have odd degree.

(e) Does G2 and/or G3 contain a cut edge? If so, identify which edge(s) are cut edges.

Solution: G3 contains a cut edge: the edge connecting p and s. G2 does not contain a cut edge.