CS 173, Fall 2012, Midterm 1 Solutions

Problem 1: Multiple choice (10 points)

Check the most appropriate box for each statement. Check only one box per statement. If you change your answer, make sure it's easy to tell which box is your final selection.

$$\sum_{k=3}^{n} k^7 =$$

$\sum_{p=1}^{n-2}(p+2)^7$ ☑ $\sum_{p=1}^{n-2}p^9$ ☐

$\sum_{p=1}^{n-2}k^7$ ☐ $\sum_{p=1}^{n-2}k^9$ ☐

For all real numbers x,
if $x^2 \leq -3$, then $x < 10$

True ☑   **False** ☐

zero is

even ☑   odd ☐

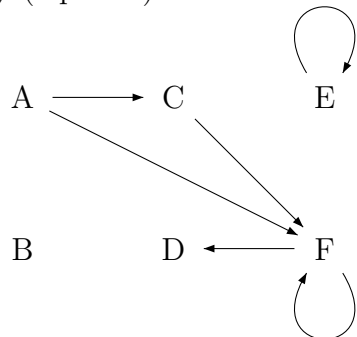both ☐   neither ☐

$\{4,5,7\} \cap \{7,8,9\} =$

7 ☐ $\{7\}$ ☑

$\{4,5,8,9\}$ ☐ $\{4,5,7,8,9\}$ ☐

$|A \cup B| = |A| + |B|$

true for any sets A and B ☐

false for any sets A and B ☐

true for some sets A and B ☑

Problem 2: Short answer (13 points)

(a) (5 points) Check all boxes that correctly characterize this relation on the set $\{A, B, C, D, E, F\}$

Reflexive: ☐   **Irreflexive:** ☐

Symmetric: ☐   **Antisymmetric:** ☑

Transitive: ☐

(b) (3 points) Suppose that $f : \mathbb{Z} \to \mathbb{R}$ is defined by $f(n) = 3n$. Identify clearly (e.g. $\mathbb{C}$, {powers of two}) the key sets in this definition.

Solution:

domain: $\mathbb{Z}$

co-domain: $\mathbb{R}$

image: {multiples of 3}

(c) (5 points) Use the Euclidean algorithm to compute $\gcd(1702, 1221)$. Show your work.

Solution:

remainder(1702,1221) = 481

remainder(1221,481) = 259

remainder(481,259) = 222

remainder(259,222) = 37

remainder(222,37) = 0

So $\gcd(1702, 1221) = 37$.

Problem 3: Number Theory (12 points)

(a) (6 points) In $\mathbb{Z}_{11}$, find the value of $[6]^6 + [5]^3$. You must show your work, keeping all numbers in your calculations small. **You may not use a calculator.** You must express your final answer as $[n]$, where $0 \leq n \leq 10$.

Solution:

$$[6]^2 = [36] = [3]$$
$$[6]^6 = ([6]^2)^3 = [3]^3 = [3^3] = [27] = [5]$$
$$[5]^3 = [5][25] = [5][3] = [15] = [4]$$
$$[6]^6 + [5]^3 = [5] + [4] = [9]$$

(b) (6 points) Let a and b be integers, $b > 0$. The "Division Algorithm" uses two formulas to define the quotient q and the remainder r of a divided by b. State these two formulas.

Solution:

$$a = bq + r$$

$$0 \leq r < b$$

Each equation is worth 3 points. This question is harder than it looks. It is very common to entirely leave out the second equation.

Problem 4: Sets (13 points)

(a) (8 points) Let $A = \{(x, y) \in \mathbb{R}^2 \mid y = \frac{2}{6}x - 3\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid x \geq 3y\}$. Prove that $A \subseteq B$.

Solution: Suppose $p \in A$. Then by the definition of A, $p = (x, y)$, where x and y are real numbers and $y = \frac{2}{6}x - 3$. Then $3y = x - 9$. So $x = 3y + 9$. But $3y + 9 \geq 3y$. So $x \geq 3y$. So $p = (x, y)$ is in B.

(b) (5 points) Suppose we have the following sets:
$$M = \{\text{cereal}, \text{toast}\}$$
$$N = \{\text{milk}, \text{coffee}, \text{wine}, \text{juice}\}$$
$$P = \{\text{wine}, \text{beer}, (\text{coffee}, \text{ham})\}$$

List the elements of $M \times (N - P)$

Solution:

$$N - P = \{\text{milk}, \text{coffee}, \text{juice}\}$$

$$M \times (N - P) = \{(\text{cereal}, \text{milk}), (\text{toast}, \text{milk}), (\text{cereal}, \text{coffee}), (\text{toast}, \text{coffee}), (\text{cereal}, \text{juice}), (\text{toast}, \text{juice})\}$$

3

Problem 5: Proofs/logic (10 points)

(a) (5 points) Suppose that J is the set of open intervals of the real line, i.e

$$J = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$$

Let's define the "touches" relation T on J by $(a, b)T(c, d)$ if and only if $a = d$ or $b = c$. Is T transitive? Informally explain why it is, or give a concrete counter-example showing that it is not.

Solution: T is not transitive. For example, $(0, 1)T(1, 2)$ and $(1, 2)T(2, 3)$ but it's not the case that $(0.1)T(2, 3)$.

(b) (5 points) State the contrapositive of the following claim. Your answer should be in words, with all negations (e.g. "not") on individual predicates.

> For all hyperreal numbers x and y, if x is floppy and y is typical, then x is acidic or $x + y$ is bubbly.

Solution: For all hyperreal numbers x and y, if x is not acidic and $x + y$ is not bubbly. then x is not floppy or y is not typical,

Problem 6: Number Theory Proof (12 points)

Congruence mod k can be defined as follows: if a, b, k are integers, k positive, then $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some integer n. Using this definition and our normal definition of $m \mid n$, prove the following claim. Use your best mathematical style, e.g. introduce variables and assumptions, justify important or non-obvious steps, put your steps in logical order.

> Claim: For any integers a, b, c, k, q, where k and q are positive, if $a \equiv b \pmod{k}$ and $b \equiv c \pmod{q}$ and $k \mid q$, then $a \equiv c \pmod{k}$.

Solution: Let a, b, c, k, q be integers. Suppose that k and q are positive, $a \equiv b \pmod{k}$, $b \equiv c \pmod{q}$, $k \mid q$.

By the definition of congruence, $a \equiv b \pmod{k}$ means that $a = b + nk$ where n is an integer. Similarly, $b \equiv c \pmod{q}$ means that $b = c + mq$, where m is an integer. Combining these two equations, we get $a = (c + mq) + nk$.

By the definition of divides, $k \mid q$ means that $q = tk$, where t is an integer. Substituting this into the previous equation, we get $a = (c + mtk) + nk$. So $a = c + (mt + n)k$. Since m, t, and n are integers, so is $mt + n$. So $a \equiv c \pmod{k}$.