CS 173: Discrete Structures, Fall 2012
Final review solutions

These problems should not be turned in. They are to help you review for the final.

1. **Counting problems**

   (a) Let $M = \{a, b, c, d\}$. How many different partitions of M are there?
   Solution: There are 15 different ways to partition M:

   - All the elements together $\{\{a, b, c, d\}\}$ (1 choice)
   - Three elements together and one separate, e.g. $\{\{a\}, \{b, c, d\}\}$. (4 choices)
   - Two pairs of elements e.g. $\{\{a, b\}, \{c, d\}\}$. (3 choices)
   - A pair of elements and two separate elements e.g. $\{\{a\}, \{b\}, \{c, d\}\}$. (6 choices)
   - All four elements separate $\{\{a\}, \{b\}, \{c\}, \{d\}\}$ (1 choice)

   (b) In the game Tic-tac-toe (played on the usual 3 by 3 grid), how many different board configurations are possible after four moves (i.e. two moves by each player)?
   Solution: There are $\binom{9}{2}$ ways to pick a set of positions to contain the X's. Given that choice, there are $\binom{7}{2}$ ways to pick two more positions to contain the O's. So the total number of configurations is $\binom{9}{2}\binom{7}{2} = \frac{9!}{2!2!5!}$. Or you can think of this as the number of ways to rearrange 2 X's 2 O's and 5 blanks in 9 spots so $\binom{9}{2\ ,\ 2} = \binom{9}{2\ ,\ 5}$.

   (c) Suppose we have a state diagram with n states and k different actions. In how many different ways could we construct a transition function for this diagram?
   Solution: The domain of the transition function contains all the pairs of a state plus an action, so it has size nk. The co-domain of the function contains all sets of states, so it has size 2^n. We can choose the output value independently for each input value, so we have $(2^n)^{nk}$ total choices for constructing the transition function.

2. **Algorithms**

   Consider the following procedure Gnarly, which returns true or false. Notice the indenting on lines 04-05: they execute only when the test on line 03 succeeds. You can assume that $n \geq 1$ and that extracting a subarray (e.g. in the recursive calls in line 06) requires only constant time.

```
01 Gnarly (a_1, ..., a_n: array of integers)
02     if (n = 1) return true
03     else if (n = 2)
04         if (a_1 = a_2) return true
05         else return false
06     else if (Gnarly(a_1, ..., a_{n-1}) and Gnarly(a_2, ..., a_n)) return true
07     else return false
```

(a) If Gnarly returns true, what must be true of the values in the input array? Briefly justify your answer.

Solution: All the values in the input array must be equal. The base cases (lines 02-05) check that that is true for very small arrays. If the array has more than three elements, then verifying that all the elements are equal when you remove each of two different elements (line 06) forces all the elements to be equal in the whole array.

(b) Express $T(n)$, the running time of Gnarly on an input of length n, as a recurrence. Be sure to include an initial condition.

Solution:

$$\begin{aligned} T(1) &= T(2) = c \\ T(n) &= 2T(n-1) + d \end{aligned}$$

(c) Give a big-theta bound on the running time of Gnarly. For full credit, you must show work or briefly explain your answer.

Solution: Gnarly takes $\Theta(2^n)$ time. This recurrence is basically the same as that for the Towers of Hanoi solver. Each node in the recursion tree contains only constant work, but there are $2^{n+1} - 1$ nodes in the tree.

3. **Counting**

(a) Suppose a set S has 11 elements. How many subsets of S have an even number of elements? Express your answer as a summation. (You do not need to simplify expressions involving permutations and/or combinations.)

Solution: Subsets of S having an even number of elements would have $0, 2, 4, 6, 8$ or 10 elements. There are $C(11, 0)$ subset of S with no elements, $C(11, 2)$ subsets of S with 2 elements, and so on. So the number of subsets of S having an even number of elements is

$$\binom{11}{0} + \binom{11}{2} + \binom{11}{4} + \binom{11}{6} + \binom{11}{8} + \binom{11}{10} = \sum_{i=0}^{5} \binom{11}{2i}.$$

(b) If $x, y, z \in \mathbb{N}$, how many solutions are there to the equation $x + y + z = 25$?

Solution: Imagine that the number 25 represents 25 objects that can be chosen in three different types: type x, type y, and type z. To indicate type we divide the objects into three bins, i.e., place 3 dividers into the list of objects. The number of ways to place the objects into the bins is the number of different solutions and is given by the expression for combinations with repetition

$$\binom{25 + 3 - 1}{25} = \binom{27}{25} = \binom{27}{2} = 351$$

(c) Suppose that A is a set containing p elements and B is a set containing n elements. How many functions are there from A to $\mathbb{P}(B)$? How many of these functions are one-to-one?

Solution: $\mathbb{P}(B)$ contains 2^n elements. So the total number of functions from A to $\mathbb{P}(B)$ is $(2^n)^p$. The number of one-to-one functions is $\frac{2^n!}{(2^n-p)!}$.

4. **Binomial theorem**

(a) How many terms are contained in $(x+y+z)^{30}$ after carrying out all multiplications, but before collecting like terms?

Solution: Note that $(x+y+z)^0 = 1$ which has $1 = 3^0$ terms. Similarly, $(x+y+z)^1 = x + y + z$ which has $3 = 3^1$ terms. Each time we increase the exponent by one, we are multiplying each of the terms from the previous expression by each of x, y, and z tripling the total number of terms. Thus, $(x + y + z)^{30}$ has 3^{30} terms.

(b) How many terms are contained in $(x+y+z)^{30}$ after carrying out all multiplications and collecting like terms?

Solution: After performing the multiplications and collecting similar terms, we are left with a sum of terms of the form $C \cdot x^i y^j z^k$, where $C, i, j, k \in \mathbb{N}$ and $i+j+k = 30$. Think of each term as having exactly 30 slots for variables (all to the first power) where we can choose from a list of three variables. Then the exponents i, j, k tell you the number of copies of x, y, z (respectively) that show up in each term. The number of ways to choose a string of length 30 from the alphabet list $\{x, y, z\}$ with repetition where order does not matter (since multiplication is commutative) is

$$\binom{30+3-1}{30} = \binom{32}{30} = \frac{32!}{30! \cdot (32-20)!} = \frac{32 \cdot 31}{2} = 496.$$

(c) What is the coefficient of the $x^{15}y^6z^9$ term? (You may leave you answer in terms of factorials!)

Solution: The coefficient is the number of permutations with repetition of a list of elements containing 15 x's, 6 y's, and 9 z's. This number is given by

$$\binom{30}{15\ ,\ 6} = \binom{30}{15} \cdot \binom{15}{6} \cdot \binom{9}{9} = \frac{30!}{15! \cdot 6! \cdot 9!}.$$

5. **Power sets**

Suppose you were given the following sets:

$$\begin{aligned}
\mathbf{A} &= \{\text{Vine}, \text{Tree}, \text{Shrub}\} \\
\mathbf{B} &= \{\{\text{Tree}\}\} \\
\mathbf{C} &= \{\text{Vine}, \text{Moss}\} \\
\mathbf{D} &= \{\text{Red}, \text{Green}\} \\
\mathbf{E} &= \{\text{Red}\}
\end{aligned}$$

For the following expressions, list the elements of the set or give the cardinality (as appropriate):

(a) $D \times C$

(b) $\mathbb{P}(B \cup E)$

(c) $|A \times \ \mathbb{P}(A \cup D)\ |$

(d) $\{S \in \mathbb{P}(A \cup D)\ :\ |S| \text{ is a multiple of } 4\}$

(e) $\mathbb{P}(\mathbb{P}(\mathbb{P}(\emptyset)))$

Solution:

(a) $D \times C = \{(\text{Red}, \text{Vine}), (\text{Red}, \text{Moss}), (\text{Green}, \text{Vine}), (\text{Green}, \text{Moss})\}$

(b) $\mathbb{P}(B \cup E) = \{\emptyset, \{\text{Red}\}, \{\{\text{Tree}\}\}, \{\{\text{Tree}\}, \text{Red}\}\}$

(c) $|A \times \ \mathbb{P}(A \cup D)\ | = |A| \cdot |\mathbb{P}(A \cup D)| = 3 \cdot 2^5 = 96$

(d) $\{S \in \mathbb{P}(A \cup D)\ :\ |S| \text{ is a multiple of } 4\} = \{\emptyset, \{\text{Vine}, \text{Tree}, \text{Shrub}, \text{Red}\},$
$\{\text{Vine}, \text{Tree}, \text{Shrub}, \text{Green}\}, \{\text{Tree}, \text{Shrub}, \text{Green}, \text{Red}\},$
$\{\text{Vine}, \text{Shrub}, \text{Green}, \text{Red}\}, \{\text{Vine}, \text{Tree}, \text{Green}, \text{Red}\}\}$

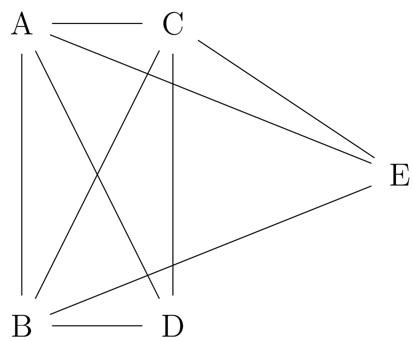
(e) $\mathbb{P}(\mathbb{P}(\mathbb{P}(\emptyset))) = \mathbb{P}(\mathbb{P}(\{\emptyset\})) = \mathbb{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

6. **Planar graphs**

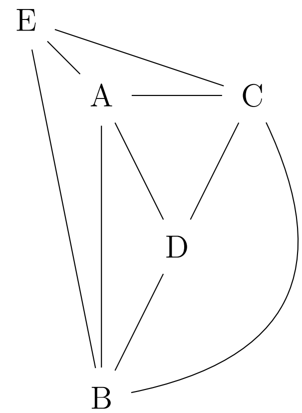
(a) Suppose that G is an undirected connected simple planar graph with 10 vertices, all of degree 4. How many edges does it have? Use Euler's formula and the Handshaking Theorem to calculate how many regions it has.

Solution: The sum of the vertex degrees is 40. By the Handshaking Theorem, there must be half that many edges, i.e. 20 edges. Euler's formula says that $v - e + f = 2$, where e is the number of edges, v the number of vertices, and f the number of regions. So, in our case, $10 - 20 + f = 2$. So there must be 12 regions.

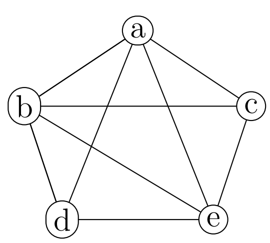
(b) Show that this graph is planar by redrawing it without crossings:



Solution:



(c) Show that the following graph is planar by redrawing it so that no two edges cross each other.



Solution: