Sets

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These notes cover set notation, operations on sets, and how to prove claims involving sets (Rosen sections 2.1 and 2.2). They also cover some logic subtleties that we didn't discuss earlier in the term: vacuous truth and the meaning of nested quantifiers. See Rosen sections 1.3 and 1.4.

1 Sets

I'm sure you've all seen the basic ideas of sets and, indeed, we've been using some of them all term. It's time to discuss sets systematically, showing you a useful range of constructions, notation, and special cases. A few operations (e.g. power sets and Cartesian products) are probably unfamiliar to many of you. And we'll see how to do proofs of claims involving sets.

Definition: A set is an unordered collection of objects.

For example, the natural numbers are a set. So are the integers between 3 and 7 (inclusive). So are all the planets in this solar system or all the programs written by students in CS 225 in the last three years. The objects in a set can be anything you want.

The items in the set are called its elements or members. We've already seen the notation for this: $x \in A$ means that x is a member of the set A.

There's three basic ways to define a set:

- describe its contents in mathematical English, e.g. "the integers between 3 and 7, inclusive."
- list all its members, e.g. $\{3, 4, 5, 6, 7\}$
- use so-called set builder notation, e.g. $\{x \in \mathbb{Z} \mid 3 \leq x \leq 7\}$

Set builder notation has two parts separated with a vertical bar (or, by some writers, a colon). The first part names a variable (in this case x) that ranges over all objects in the set. The second part one or more constraints that these objects must satisfy, e.g. $3 \leq x \leq 7$. The type of the variable (integer in our example) can be specified either before or after the vertical bar. The separator ($|$ or :) is often read "such that."

Here's an example of a set containing an infinite number of objects

- "multiples of 7"
- $\{\ldots - 14, -7, 0, 7, 14, 21, 18, \ldots\}$
- $\{x \in \mathbb{Z} \mid x \text{ is a multiple of } 7\}$

We couldn't list all the elements, so we had to use "...". This is only a good idea if the pattern will be clear to your reader. If you aren't sure, use one of the other methods.

If we wanted to use shorthand for "multiple of", it might be confusing to have $|$ used for two different purposes. So it would probably be best to use the colon variant of set builder notation:

$\{x \in \mathbb{Z} : 7 \mid x\}$

2 Things to be careful about

A set is an unordered collection. So $\{1, 2, 3\}$ and $\{2, 3, 1\}$ are two descriptions of the same set.

Each element occurs only once in a set. Or, alternatively, it doesn't matter how many times you write it. So $\{1, 2, 3\}$ and $\{1, 2, 3, 2\}$ also describe the same set.

So sets are very different from ordered tuples such as $(1, 2, 2, 3)$.

Tuple? What's that? $(3, 4)$ is a pair. $(4, 5, 2)$ is a triple. $(4, 5, 2, 6)$ is a quadruple. But most people's knowledge of latinate prefixes breaks down very fast. So mathematicians often call it a 4-tuple. So a pair would be a 2-tuple. And the general class of objects are tuples.

For ordered tuples, the order of values matters and duplicate elements don't magically collapse. $(1, 2, 2, 3) \neq (1, 2, 3)$ and $(1, 2, 2, 3) \neq (2, 2, 1, 3)$. Therefore, make sure to enclose the elements of a set in curly brackets and carefully distinguish curly brackets (set) from parentheses (ordered pair).

The empty set (also called the null set) is a special set that contains no elements. It is written $\emptyset$. Don't write it as $\{\}$, even though that might seem sensible. The notation $\emptyset$ is firmly entrenched in mathematics and not using it will cause readers to think less of your mathematical skill. (It's like making a spelling mistake on your resume.)

The empty set may seem like a pain in the neck. However, computer science applications are full of empty lists, strings of zero length, and the like. It's the kind of special case that all of you (even the non-theoreticians) will spend your life having to watch out for.

A set can contain objects of more than one type, e.g. $\{a, b, 3, 7\}$. A set can also contain sets, e.g. $\{\mathbb{Z}, \mathbb{Q}\}$ is a set containing two infinite sets. $\{\{a, b\}, \{c\}\}$ is a set containing two finite sets. Notice that $\{\emptyset\}$ is not the empty set but, rather, a set with one member, which is the empty set.

If you are having trouble with these layers of structure, imagine each set to be a box. Then $\{\{a, b\}, \{c\}\}$ is a box containing two boxes. One of the inside boxes contains a and b, the other contains c. $\{\emptyset\}$ would then be a box containing a second box, with the inside box empty.

3 Cardinality, inclusion

If A is a finite set (a set containing only a finite number of objects), then $|A|$ is the number of objects in A. This is also called the **cardinality** of A. For example, $|\{a, b, 3\}| = 3$. The notation of cardinality also extends to sets with infinitely many members ("infinite sets") such as the integers, but we won't get into the details of that right now.

Notice that the notation $|A|$ might mean set cardinality or it might be the more familiar absolute value. To tell which, figure out what type of object A is. If it's a set, the author meant cardinality. If it's a number, the author meant absolute value.

If the set contains objects with complex structure, the cardinality is the number of top-level objects. Don't flatten out the structure and count the basic objects in them. For example

$$|\{1, \{1\}, \{\{1\}, 3\}\}| = 3$$

The top-level objects are

- 1
- $\{1\}$
- $\{\{1\}, 3\}$

If A and B are sets, then A is a subset of B (writte $A \subseteq B$) if every element of A is also in B. Or, if you want it formally: $\forall x, x \in A \rightarrow x \in B$. For example, $\mathbb{Q} \subseteq \mathbb{R}$, because every member of the rationals is also a member of the reals.

The notion of subset allows the two sets to be equal. So $A \subseteq A$ is true for any set A. So $\subseteq$ is like $\leq$. If you want to force the two sets to be different (i.e. like $<$), you must say that A is a **proper** subset of B, written $A \subset B$. You'll occasionally see reversed versions of these symbols to indicate the opposite relation, e.g. $B \supseteq A$ means the same as $A \subseteq B$.

4 Vacuous truth

If we have a set A, an interesting question is whether the empty set should be considered a subset of A. To answer this, let's first back up and look at one subtlety of mathematical logic.

Consider the following claim:

Claim 1 *For all natural numbers n, if $14 + n < 10$, then n wood elves will attack Siebel Center tomorrow.*

I claim this is true, a fact which most students find counter-intuitive. In fact, it wouldn't be true if n was declared to be an integer.

Notice that this statement has the form $\forall n, P(n) \rightarrow Q(n)$, where $P(n)$ is the predicate $14+n < 10$. Because n is declared to be a natural number, n is never negative, so $n + 14$ will always be at least 14. So $P(n)$ is always false. Therefore, our conventions about the truth values for conditional statements imply that $P(n) \rightarrow Q(n)$ is true. This argument works for any choice of n. So $\forall n, P(n) \rightarrow Q(n)$ is true.

Because even mathematicians find such statements a bit wierd, they typically say that such a claim is *vacuously* true, to emphasize to the reader that it is only true because of this strange convention about the meaning of conditionals. Vacuously true statements typically occur when you are trying to apply a definition or theorem to a special case involving an abnormally small or simple object, such as the empty set or zero or a graph with no arrows at all.

In particular, this means that the empty set is a subset of any set A. For $\emptyset$ to be a subset of A, the definition of "subset" requires that for every object x, if x is an element of the empty set, then x is an element of A. But this if/then statement is considered true because its hypothesis is always false.

5 Set operations

Given two sets A and B, the intersection of A and B ($A \cap B$) is the set containing all objects that are in both A and B. In set builder notation:

$$A \cap B = \{S \mid S \in A \text{ and } S \in B\}$$

Let's set up some sample sets:

- $M = \{\text{egg}, \text{bread}, \text{milk}\}$
- $P = \{\text{milk}, \text{egg}, \text{flour}\}$

Then $M \cap P$ is $\{\text{milk}, \text{egg}\}$.

If the intersection of two sets A and B is the empty set, i.e. the two sets have no elements in common, then A and B are said to be **disjoint**.

The union of sets A and B ($A \cup B$) is the set containing all objects that are in one (or both) of A and B. So $M \cup P$ is $\{\text{milk}, \text{egg}, \text{bread}, \text{flour}\}$.

The set difference of A and B ($A - B$) contains all the objects that are in A but not in B. In this case,

$$M - P = \{\text{bread}\}$$

The complement of a set A ($\overline{A}$) is all the objects that aren't in A. For this to make sense, you need to define your "universal set" (often written U). U contains all the objects of the sort(s) you are discussing. For example, in some discussions, U might be all real numbers. U doesn't contain everything you might imagine putting in a set, because constructing a set that inclusive leads to paradoxes. U is more limited than that. Whenever U is used, you and your reader need to come to an understanding about what's in it.

So, if our universe is all integers, and A contains all the multiples of 3, then $\overline{A}$ is all the integers whose remainder mod 3 is either 1 or 2. $\overline{\mathbb{Q}}$ would be the irrational numbers if our universe is all real numbers. If we had been working with complex numbers, it might be the set of all irrational real numbers plus all the numbers with an imaginary component.

6 Power sets and Cartesian products

If A is a set, the powerset of A (written $\mathbb{P}(A)$ or 2^A) is the set containing all subsets of A. For example, suppose that $A = \{1, 2, 3\}$. Then

$$\mathbb{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

If S is finite and contains n elements, then $\mathbb{P}(S)$ contains 2^n elements. Or, in shorthand, $|\mathbb{P}(S)| = 2^{|S|}$. This is the reason behind the alternate notation 2^S for the powerset of S.

Notice that the powerset of A always contains the empty set, regardless of what's in A. As a consequence, $\mathbb{P}(\emptyset) = \{\emptyset\}$.

If A and B are two sets, their Cartesian product ($A \times B$) contains all ordered pairs (x, y) where x is in A and y is in B. That is

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

For example, if $A = \{a, b\}$ and $B = \{1, 2\}$, then

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

Notice that order matters for Cartesian product.

$$B \times A = \{(1, a), (2, a), (1, b), (2, b)\}$$

These sets are typically not equal.

If $|A| = n$ and $|B| = m$, then $|A \times B| = nm$.

7 Set identities

Rosen lists a large number of identities showing when two sequences of set operations yield the same output sets. For example:

DeMorgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

I won't go through these in detail because they are largely identical to the identities you saw for logical operations, if you make the following correspondences:

- $\cup$ is like $\vee$
- $\cap$ is like $\wedge$
- $\overline{A}$ is like $\neg P$
- $\emptyset$ (the empty set) is like F
- U (the universal set) is like T

The two systems aren't exactly the same. E.g. set theory doesn't use a close analog of the $\rightarrow$ operator. But they are very similar.

8 Proving facts about set inclusion

So far in school, most of your proofs or derivations have involved reasoning about equality. Inequalities (e.g. involving numbers) have been much less common. With sets, the situation is reversed. Proofs typically involve reasoning about subset relations, even when proving two sets to be equal. Proofs that rely primarily on a chain of set equalities do occur, but they are much less common. Even when both approaches are possible, the approach based on subset relations is often easier to write and debug.

As a first example of a typical set proof, consider the following claim:

Claim 2 *For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

This property is called "transitivity," just like similar properties for (say) $\leq$ on the real numbers. Both $\subseteq$ and $\leq$ are examples of a general type of object called a *partial order*, for which transitivity is a key defining property.

First, remember our definition of $\subseteq$: a set A is a subset of a set B if and only if, for any object x, $x \in A$ implies that $x \in B$.

Proof: Let A, B, and C be sets and suppose that $A \subseteq B$ and $B \subseteq C$.

Our ultimate goal is to show that $A \subseteq C$. This is an if/then statement: for any x, if $x \in A$, then $x \in C$. So we need to pick a representative x and assume the hypothesis is true, then show the conclusion. So our proof continues:

Let x be an element of A. Since $A \subseteq B$ and $x \in A$, then $x \in B$ (definition of subset). Similarly, since $x \in B$ and $B \subseteq C$, $x \in C$. So for any x, if $x \in A$, then $x \in C$. So $A \subseteq C$ (definition of subset again). $\square$

9 Example proof: deMorgan's law

Two strategies are commonly used for proving that two sets A and B are equal. One method is to show the equality via a chain of equalities. This is great when it works but it requires each step to work in both directions.

The more common strategy is to show that $A \subseteq B$ and $B \subseteq A$, using separate subproofs. We can then conclude that $A = B$. This is just like showing that $x = y$ by showing that $x \leq y$ and $y \leq x$. Although it seems like more trouble to you right now, this is a more flexible approach that works in a wider range of situations, especially in upper-level computer science and mathematics courses (e.g. real analysis, algorithms).

As an example, let's look at

Claim (DeMorgan's Law): For any sets A and B, $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Let's prove this in both styles:

Proof 1: Let A and B be sets. Then

$$
\begin{aligned}
\overline{A \cup B} &= \{x \mid x \notin A \cup B\} \\
&= \{x \mid \text{ it's not the case that } (x \in A \text{ or } x \in B)\} \\
&= \{x \mid x \notin A \text{ and } x \notin B\} \\
&= \{x \mid x \in \overline{A} \text{ and } x \in \overline{B}\} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$

Proof 2: Let A and B be sets. We'll do this in two parts:

$\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$: Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$. So it's not the case that ($x \in A$ or $x \in B$). So, by deMorgan's Law for logic, $x \notin A$ and $x \notin B$. That is $x \in \overline{A}$ and $x \in \overline{B}$. So $x \in \overline{A} \cap \overline{B}$.

$\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$: very similar.

In this example, the second half is very basically the first half written backwards. However, this is not always the case. The power of this method comes from the fact that you can apply different techniques to proving the two subset relations.

10 An example with products

Let's move on to some facts that aren't so obvious, because they involve Cartesian products.

Claim 3 *If A, B, C, and D are sets such that $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.*

To prove this, we first gather up all the facts we are given. What we need to show is the subset inclusion $A \times C \subseteq B \times D$. To do this, we'll need to pick a representative element from $A \times C$ and show that it's an element of $B \times D$.

Proof: Let A, B, C, and D be sets and suppose that $A \subseteq B$ and $C \subseteq D$.

Let $p \in A \times C$. By the definition of Cartesian product, p must have the form (x, y) where $x \in A$ and $y \in C$.

Since $A \subseteq B$ and $x \in A$, $x \in B$. Similarly, since $C \subseteq D$ and $y \in C$, $y \in D$. So then $p = (x, y)$ must be an element of $B \times D$.

We've shown that, for all p, if $p \in A \times C$ then $p \in B \times D$. This means that $A \times C \subseteq B \times D$. □

The last paragraph is optional. When you first start, it's a useful recap because you might be a bit fuzzy about what you needed to prove. As you get experienced with this sort of proof, it's often omitted. But you will still see it occasionally at the end of a very long (e.g. multi-page) proof, where even an experienced reader might have forgotten the main goal of the proof.

11 Another example with products

Here's another claim about Cartesian products:

Claim 4 *For any sets A, B, and C, if $A \times B \subseteq A \times C$ and $A \neq \emptyset$, then $B \subseteq C$.*

Notice that this is like dividing both sides of an algebraic equation by a non-zero number: if $xy \leq xz$ and $x \neq 0$ then $y \leq z$. The claim fails if we allow x to be zero. Since the empty set plays the role of zero in set theory, this suggests why we have the analogous condition in our claim about sets. Although there are occasionally differences between sets and numbers, the parallelism is strong enough to suggest special cases that you should be sure to investigate.

A general property of proofs is that the proof should use all the information in the hypothesis of the claim. If that's not the case, either the proof has a bug (e.g. on a CS 173 homework) or the claim could be revised to make it more interesting (e.g. when doing a research problem, or a buggy homework problem). Either way, there's an important issue to deal with. So, in this case, we need to make sure that our proof does use the fact that $A \neq \emptyset$.

Here's a draft proof:

Proof draft 1: Suppose that A, B, C, and D are sets and suppose that $A \times B \subseteq A \times C$ and $A \neq \emptyset$. We need to show that $B \subseteq C$.

So let's choose some $x \in B$. ...

The main fact we've been given is that $A \times B \subseteq A \times C$. To use it, we need an element of $A \times B$. Right now, we only have an element of B. We need to find an element of A to pair it with. To do this, we reach blindly into A, pull out some random element, and give it a name. But we have to be careful here: what if A does contain any elements? So we have to use the assumption that $A \neq \emptyset$.

Proof draft 1: Suppose that A, B, C, and D are sets and suppose that $A \times B \subseteq A \times C$ and $A \neq \emptyset$. We need to show that $B \subseteq C$.

So let's choose some $x \in B$. Since $A \neq \emptyset$, we can choose an element t from A. Then $(t, x) \in A \times B$ by the definition of Cartesian product.

Since $(t, x) \in A \times B$ and $A \times B \subseteq A \times C$, we must have that $(t, x) \in A \times C$ (by the definition of subset). But then (again by the definition of Cartesian product) $x \in C$.

So we've shown that if $x \in B$, then $x \in C$. So $B \subseteq C$, which is what we needed to show.

12 A proof using power sets

Now, we can prove a claim about power sets:

Claim 5 *For all sets A and B, $A \subseteq B$ if and only if $\mathbb{P}(A) \subseteq \mathbb{P}(B)$.*

Our claim is an if-and-only-if statement. The normal way to prove such a statement is by proving the two directions of the implication separately. Although it's occasionally possible to do the two directions together, this doesn't always work and is often confusing to the reader.

Proof ($\rightarrow$): Suppose that A and B are sets and $A \subseteq B$. Suppose that S is an element of $\mathbb{P}(A)$. By the definition of power set, S must be a subset of A. Since $S \subseteq A$ and $A \subseteq B$, we must have that $S \subseteq B$ (by transitivity). Since $S \subseteq B$, the definition of power set implies that $S \in \mathbb{P}(B)$. Since we've shown that any element of $\mathbb{P}(A)$ is also an element of $\mathbb{P}(B)$, we have that $\mathbb{P}(A) \subseteq \mathbb{P}(B)$.

A really common mistake is to stop at this point, thinking you are done. But we've only done half the job. We need to show that the implication works in the other direction:

Proof ($\leftarrow$): Suppose that A and B are sets and $\mathbb{P}(A) \subseteq \mathbb{P}(B)$. Notice that $A \subseteq A$. By the definition of power set, this implies that $A \in \mathbb{P}(A)$. Since $A \in \mathbb{P}(A)$ and $\mathbb{P}(A) \subseteq \mathbb{P}(B)$, we know that $A \in \mathbb{P}(B)$ (definition of subset). So, by the definition of power set, $A \subseteq B$. $\square$

13 A proof using sets and contradiction

Here's a claim about sets that's less than obvious:

Claim 6 *For any sets A and B, if $(A-B) \cup (B-A) = A \cup B$ then $A \cap B = \emptyset$.*

Notice that the conclusion $A \cap B = \emptyset$ claims that something does not exist (i.e. an object that's in both A and B). So this is a good place to apply proof by contradiction.

Proof: Suppose not. That is, suppose that A and B are sets, $(A-B) \cup (B-A) = A \cup B$ but $A \cap B \neq \emptyset$.

Since $A \cap B \neq \emptyset$, we can choose an element from $A \cap B$. Let's call it x.

Since x is in $A \cap B$, x is in both A and B. So x is in $A \cup B$.

However, since x is in B, x is not in $A - B$. Similarly, since x is in A, x is not in $B - A$. So x is not a member of $(A - B) \cup (B - A)$. This means that $(A - B) \cup (B - A)$ and $A \cup B$ cannot be equal, because x is in one but not the other. This contradicts our assumption at the start of the proof.

Therefore $A \cap B$ must be empty. $\square$.

14 Quantifier scope

Before we move onto functions, we need to digress a bit and fill in some facts about quantifiers.

As an example, I claim that $\mathbb{P}(A \cup B)$ is not (always) equal to $\mathbb{P}(A) \cup \mathbb{P}(B)$. We can disprove this claim with a concrete counter-example. Suppose that $A = \{x, y\}$ and $B = \{y, z\}$. Then $\{x, y, z\}$ is in $\mathbb{P}(A \cup B) = \mathbb{P}(\{x, y, z\})$. But it's not in $\mathbb{P}(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$, nor in $\mathbb{P}(B) = \{\emptyset, \{z\}, \{y\}, \{z, y\}\}$.

Let's look at this using quantifiers. Suppose that X is some set. $X \in \mathbb{P}(A)$ if $X \subseteq A$ i.e. $\forall x \in X, x \in A$. So $X \in \mathbb{P}(A) \cup \mathbb{P}(B)$ if

$$(\forall x \in X, x \in A) \vee (\forall y \in X, y \in B)$$

A quantifier is said to *bind* its variable. And the *scope* of that binding is the portion of equations and/or text during which that binding is supposed to be in force. Normally, the scope extends to the end of the sentence, unless the variable is redefined by a new quantifier. In this example, the scope of the binding of x is the whole statement. Or, you could say the scope is just the first half of the statement, since x is never used in the second half.[1] The scope of the binding of y is the second half.

A more sloppy or distracted author might write this with two copies of the variable x.

$$(\forall x \in X, x \in A) \vee (\forall x \in X, x \in B)$$

[1]It doesn't really matter which way you want to think about such cases.

This actually means the same thing. There's two different bindings for x, once of which lasts ("has scope") for the first half of the statement and one of which has scope over the second half. Sometimes you have to look carefully at parentheses to figure out how long the author intended a variable binding to last.

Now, let's look at $X \in \mathbb{P}(A \cup B)$. This is the case if

$$\forall x \in X, x \in A \vee x \in B$$

This time, there's only one variable binding, extending for the whole sentence.

The two versions don't mean the same thing. The statement $\forall x \in X, x \in A \vee x \in B$ requires that every x belongs to one of the two sets. The statement $(\forall x \in X, x \in A) \vee (\forall y \in X, y \in B)$ requires that either all the values satisfy a more restrictive condition (belonging to A) or that all the values satisfy a second more restrictive condition (belonging to B).

15 Nested quantifiers

The more interesting cases arise when we set up two quantified variables and then use a predicate that refers to both variables at once. These are called *nested quantifiers*. For example,

> For every person p in the Fleck family, there is a toothbrush t such that p brushes their teeth with t.

This sentence asks you to consider some random Fleck. Then, given that choice, it asserts that they have a toothbrush. The toothbrush is chosen after we've picked the person, so the choice of toothbrush can depend on the choice of person. This doesn't absolutely force everyone to pick their own toothbrush. (For a brief period, two of my sons were using the same one because they got confused.) However, at least this statement is consistent with each person having their own toothbrush.

Suppose now that we swap the order of the quantifiers, to get

> There is a toothbrush t, such that for every person p in the Fleck family, p brushes their teeth with t.

In this case, we're asked to choose a toothbrush t first. Then we're asserting that every Fleck uses this one fixed toothbrush t. Eeeuw!

We'd want the quantifiers in this order when there's actually a single object that's shared among the various people, as in:

> There is a stove s, such that for every person p in the Fleck family, p cooks his food on s.

When you try to understand or prove a statement with nested quantifiers, think of making a sequence of choices for the values, one after another.

Notice that a statement with multiple quantifiers is only difficult to understand when it contains a mixture of existential and universal quantifiers. If all the quantifiers are existential, or if all the quantifiers are universal, the order doesn't matter and the meaning is usually what you'd think.

16 Nested quantifiers in mathmatics

Suppose that S is a set of real numbers, then a real number x is called an *upper bound* for S if

$$\forall y \in S, y \leq x$$

For example, 2.5 is an upper bound for the set $A = \{-3, 1.5, 2\}$. So is 2. So is 3.14159.

S is *bounded above* if there is some upper bound for S, i.e.

$$\exists x \in \mathbb{R}, \forall y \in S, y \leq x$$

For example, our set A is bounded above. But the set of even integers is not.

Notice that the existential quantifier came first, so we are requiring one choice of x (the upper bound) to work for all elements of S. This is the shared stove case.

What if we reverse the order of the quantifiers, to get:

$$\forall y \in S, \exists x \in \mathbb{R}, y \leq x$$

This is the personal toothbrush case. For each element of x, we're claiming that there is a larger real number. That's obviously true, because we could just pick $x + 1$. So this weaker condition is true for any set of real numbers, even ones that stretch off to infinity like the even integers.