Planar Graphs I

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This lecture surveys facts about graphs that can be drawn in the plane without any edges crossing (first half of section 9.7 of Rosen).

1 Planar graphs

So far, we've been looking at general properties of graphs and very general classes of relations. Today, we'll concentrate on a limited class of graph: undirected connected graphs. And connected means that there's a path between any two vertices. And we assume (without ever saying this explicitly) that all graphs are finite.

Which of these graphs are "planar" i.e. can be drawn in the plane without any edges crossing (i.e. not at a vertex)?

Examples: K_4 is planar, cube (Q_3) is planar, $K_{3,3}$ isn't. See pictures in Rosen p. 658.

Notice that some pictures of a planar graph may have crossing edges. What makes it planar is that you can draw at least one picture of the graph with no crossings.

Why should we care? Connected to a variety of neat results in mathematics. (I'll show one today.) Also, crossings are a nuisance in practical design problems for circuits, subways, utility lines. Two crossing connections normally means that the edges must be run at different heights. This isn't a big issue for electrical wires, but it creates extra expense for some types of lines e.g. burying one subway tunnel under another (and therefore deeper than you would ordinarily need). Circuits, in particular, are easier to manufacture if their connections live in fewer layers.

2 Euler's formula

A planar graph with cycles divides the plane into a set of regions, also called *faces*. Each region is bounded by a simple cycle of the graph: the path bounding each region starts and ends at the same vertex and uses each edge only once. Notice that, by convention, we also count the unbounded area outside the whole graph as one region.

Examples: a cycle (2 regions), a figure 8 graph (3 regions).

This neat division of the plane into a set of regions seems intuitively obvious, but actually depends on a result from topology called the "Jordan curve theorem" which states that any simple closed curve (i.e. doesn't cross itself, starts and ends at the same place) divides the plane into exactly two regions. Proving this theorem requires worrying about the possibility that the curve has infinitely complex patterns of maze-like wiggles, but we won't go there.

Suppose that G is a connected planar graph, with v vertices, e edges, and f faces. Then Euler's formula states that:

$$v - e + f = 2$$

3 Trees

Before we try to prove Euler's formula, let's look at one special type of planar graph: trees. In graph theory, a tree is any connected graph with no cycles. When we normally think of a tree, it has a designated root (top) vertex. In graph theory, these are called *rooted trees*. For what we're doing this class, we don't need to care about which vertex is the root.

A tree doesn't divide the plane into multiple regions, because it doesn't contain any cycles. In graph theory jargon, a tree has only one face: the entire plane surrounding it. So Euler's theorem reduces to $v - e = 1$, i.e. $e = v - 1$. Let's prove that this is true, by induction.

> Proof by induction on the number of edges in the graph.
>
> Base: If the graph contains no edges and only a single vertex, the formula is clearly true.
>
> Induction: Suppose the formula works for all trees with up to n vertices. Let T be a tree with $n + 1$ vertices. We need to show that T has n edges.
>
> Now, we find a vertex with degree 1 (only one edge going into it). To do this start at any vertex r and follow a path in any direction, without repeating edges. Because T has no cycles, this path can't return to any vertex it has already visited. So it must eventually hit a dead end: the vertex at the end must have degree 1. Call it p.
>
> Remove p and the edge coming into it, making a new tree T' with n vertices. By the inductive hypothesis, T' has $n - 1$ edges. So T has n edges. Therefore the formula holds for T.

4 Proof of Euler's formula

We can now prove Euler's formula ($v - e + f = 2$) works in general, for any connected planar graph.

> Proof: by induction on the number of edges in the graph.
>
> Base: If $e = 0$, the graph consists of a single vertex with a single region surrounding it. So we have $1 - 0 + 1 = 2$ which is clearly right.
>
> Induction: Suppose the formula works for all graphs with no more than n edges. Let G be a graph with $n + 1$ edges.
>
> Case 1: G doesn't contain a cycle. So G is a tree and we already know the formula works for trees.

Case 2: G contains at least one cycle. Pick an edge p that's on a cycle. Remove p to create a new graph G'.

Since the cycle separates the plane into two regions, the regions to either side of p must be distinct. When we remove the edge p, we merge these two regions. So G' has one fewer regions than G.

Since G' has n edges, the formula works for G' by the induction hypothesis. That is $v' - e' + f' = 2$. But $v' = v$, $e' = e - 1$, and $f' = f - 1$. Substituting, we find that

$$v - (e - 1) + (f - 1) = 2$$

So

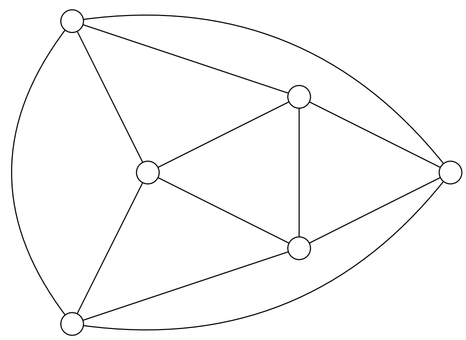
$$v - e + f = 2$$

5 Application: platonic solids

A fact dating back to the Greeks is that there are only five *platonic solids*. These are convex polyhedra whose faces all have the same number of sides (k) and whose vertices all have the same number of edges going into them (d).

Show a picture of the five platonic solids from the web: cube, dodecahedron, tetrahedron, icosahedron, octahedron, e.g. wikipedia "platonic solids".

To turn a platonic solid into a graph, imagine that it's made of a stretchy material. Make a small hole in one face. Put your fingers into that face and pull sideways, stretching that face really big and making the whole thing flat. For example, an octahedron (8 triangular sides) turns into the following graph. Notice that it still has eight regions, one for each face of the original solid, each with three sides.



Graphs of polyhedra are slightly special planar graphs. Polyhedra aren't allowed to have extra vertices partway along edges, so each vertex in the graph must have degree at least three. Also, since the faces must be flat and the edges straight, each face needs to be bounded by at least three edges. The number of edges bounding each face is called the degree of the face, so in graph theory jargon, we say that each face has degree at least three.

To be the graph of a platonic solid, all the vertices in the graph must have the same degree $d \geq 3$ and all faces must have the same degree $k \geq 3$. I claim that the graphs of the five platonic solids are the only planar graphs which satisfy these conditions.

Proof: By the handshaking theorem, the sum of the vertex degrees is twice the number of edges. So, since the degrees are equal to d, we have

$$dv = 2e$$

Each edge is on the boundary of two regions. So the sum of the region degrees is also twice the number of edges. That is

$$kf = 2e$$

So this means that $v = \frac{2e}{d}$ and $f = \frac{2e}{k}$

Euler's formula says that $v - e + f = 2$. Substituting into this, we get:

$$\frac{2e}{d} - e + \frac{2e}{k} = 2$$

So

$$\frac{2e}{d} + \frac{2e}{k} = 2 + e$$

Dividing both sides by $2e$:

$$\frac{1}{d} + \frac{1}{k} = \frac{1}{e} + \frac{1}{2}$$

If we analyze this equation, we discover that d and k can't both be larger than 3. If they are both 4 or above, the left side of the equation is at most $\frac{1}{2}$. But since e is positive, the righthand side of the equation must be larger than $\frac{1}{2}$. So one of d and k is actually equal to three and the other is some integer that is at least 3.

Suppose we set d to be 3. Then the equation becomes

$$\frac{1}{3} + \frac{1}{k} = \frac{1}{e} + \frac{1}{2}$$

So

$$\frac{1}{k} = \frac{1}{e} + \frac{1}{6}$$

Since $\frac{1}{e}$ is positive, this means that k can't be any larger than 5.

Similarly, if k is 3, then d can't be any larger than 5.

This leaves us only five possibilities for the degrees d and k: $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$, and $(5, 3)$.

Once we've pinned down the degrees of all the vertices in the graph, we've pinned down the basic structure of the graph and of the corresponding solid figure. So there are only five possible graph structures and thus five possible platonic solids.

At several points in this proof, it's probably not obvious why you would make that step e.g. in the algebra. This is the kind of proof that would have been constructed by trying several ideas and fiddling around with the algebra and the real-world geometrical problem. It's the kind of thing mathematicians do when stuck in the back of a boring committee meeting.