Applications of Equivalence Relations

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This lecture finishes the topic of equivalence relations (section 8.5 of Rosen), showing more applications of equivalence relations and equivalence classes. It's a half-lecture due to the third quiz.

1 Announcements

Remember to check your final exam schedule for conflicts. Conflicts must be reported to the instructors no later than the last day of classes. Notice that issues near the end of term are regulated by a fairly strict set of rules, which instructors don't have much ability to modify. In particular, if you get sick towards the end of the term, it's vital to involve the emergency dean because only they can authorize extensions past the end of the term.

There will be a makeup quiz at the start of class next Monday. This can be used to substitute for an unexcused missed quiz or a really poor quiz grade. However, scores from the makeup quiz are discounted 20%, so there's no point in taking it if you have three semi-ok quiz grades or if I've already promised to interpolate a score for an excused missed quiz.

2 Rays in space

Let's see another, more complex, example of an equivalence relation. Specifically, let's model the set of rays from the origin out to infinity in 3D. Con-

structions similar to this are often used to model physical systems such as cameras.

Let $A = \mathbb{R}^3$. That is, A is 3D space. We would like to treat two points as equivalent if they lie on the same ray. That is:

> $(x, y, z) \sim (p, q, r)$ if and only if there is a positive real number λ such that $(x, y, z) = \lambda(p, q, r)$ i.e. $x = \lambda p$, $y = \lambda q$, and $z = \lambda r$.

Each equivalence class of $\sim$ is a ray from the origin, except that the origin is in a separate equivalence class all by itself. For example, $[(3, 2, 7)]$ is the ray passing through the point $(3, 2, 7)$ and it contains other points like $(1.5, 1, 3.5)$ and $(3.75, 2.5, 8.75)$.

Let's prove that $\sim$ is an equivalence relation. [In lecture, I'll probably only actually do transitivity.]

Reflexive: If $\lambda = 1$, then $(x, y, z) = \lambda(x, y, z)$. So $(x, y, z) \sim (x, y, z)$.

Symmetric: Suppose $(x, y, z) \sim (p, q, r)$ Then there is a positive real λ such that $(x, y, z) = \lambda(p, q, r)$. But then $(p, q, r) = \frac{1}{\lambda}(x, y, z)$. So $(p, q, r) \sim (x, y, z)$. If λ is a positive real, so is $\frac{1}{\lambda}$.

Transitive: Suppose that $(x, y, z) \sim (p, q, r)$ and $(p, q, r) \sim (a, b, c)$. Then there are positive reals λ and κ such that $(x, y, z) = \lambda(p, q, r)$ and $(p, q, r) = \kappa(a, b, c)$. Substituting the second equation into the first, we get that $(x, y, z) = \lambda\kappa(a, b, c)$. Since λ and κ are positive reals, so is $\lambda\kappa$. So by the definition of $\sim$, $(x, y, z) \sim (a, b, c)$.

Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation.

3 Defining real numbers via equivalence classes

Now, suppose that we know about the integers (and maybe the rational numbers) and we would like to define the real numbers. One way to do this is using equivalence classes. To make this simple, let's just worry about defining the real numbers between 0 and 1. If you can do that, it's not hard to define the full set of real numbers.

Suppose $D = \{0, 1, 2, \ldots, 9\}$ is the set of decimal digits. Let's make a set S containing all infinite sequences of digits. Each sequence will represent the decimal representation of a real. Formally, each sequence is a function from the natural numbers to the digits. That is, $S = \{f \text{ a function } \mid f : \mathbb{N} \to D\}$

So S is the set of real numbers, right? Well, sort of. The difficulty is that many reals have two representations, one ending in an infinite sequence of zeros and one ending in an infinite sequence of nines. So, we need to set up an equivalence relation $\sim$ that relates each sequence ending in an infinite string of nines with the sequence that contains the next larger number followed by an infinite sequence of zeros. $0.8789999\ldots \sim 0.879000\ldots$. The real numbers between zero and one are then the equivalence classes of this relation $\sim$, i.e. we merge each pair of equivalent decimals into a single object.

4 Mathematical glue

We can also use equivalence classes to cut-and-paste geometrical objects. For example, consider the interval $I = [0, 1]$ of the real line. You can think of it like a piece of string. If we were to glue together the two ends of the string, we would get a little loop or circle of string.

To do this mathematically, we create an equivalence relation S in which most points are in their own one-element equivalence class, but 0 and 1 are in the same equivalence class. That is, that xSy if $x = y$, or if $x = 0$ and $y = 1$, or $x = 1$ and $y = 0$. Then I/S is the equivalence classes of I under the relation S and I/S is a loop.

Now, let's see how to construct a Möbius strip. A physical Möbius strip is made by taking a strip of paper, giving it a half twist, and taping the ends together. This object is neat, because it has only one side (not two) and only one edge.

To define a Möbius strip mathematically, we select a rectangular piece of the real plane, e.g. $[0, 1] \times [0, 1]$, i.e. all the points with both coordinates between 0 and 1 (inclusive). Let's call this patch of 2D surface I^2.

We can define various equivalence relations on I, thereby gluing I to itself in different ways. First, suppose we define R by $(x, y)R(p, q)$ if

- $(x, y) = (p, q)$ or

- $y = q$, $x = 0$, and $p = 1$, or

- $y = q$, $x = 1$, and $p = 0$.

Then the set of equivalence classes I^2/R is mostly like I^2, except that we've glued together the left and right edges of I^2. So it's a tube.

Similarly, if we also glue the top of I^2 to the bottom, we'll get a torus. If we merge all the edge points of I^2 into one point, we'll get a sphere. We could do that with the relation Q in which $(x, y)Q(p, q)$ if and only if

- $(x, y) = (p, q)$ or

- x, y, p and q all come from the set $\{0, 1\}$

Now, suppose that we set up the following equivalence relation

- $(x, y) = (p, q)$ or

- $y = 1 - q$, $x = 0$, and $p = 1$, or

- $y = 1 - q$, $x = 1$, and $p = 0$.

This merges points on the left and right edges, but attaches the one edge upside down. Or, viewed another way, we put a half twist into the strip of surface before we glue the ends. When we merge equivalence points to create I^2/T, we have a Möbius strip.

If you carry on like this, you can build up all sorts of interesting and strange deformed surfaces, starting with pieces of the ordinary 2D plane.

Filling in the details of examples like these could be a bit tricky, beyond what we plan to put on homeworks or exams. But they illustrate why the idea of equivalence relations is useful in later mathematics and CS courses.