Equivalence Relations and Operations

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This lecture continues the topic of equivalence relations (section 8.5 of Rosen), including how to prove that something is an equivalence relation and how to show that an operation on equivalence classes is well-defined.

1 Announcements

Reminder: quiz Wednesday. Studying for that will probably be your big agenda item for the next couple days, especially since no one properly absorbs the last couple lectures before Thanksgiving break.

Honors homework 4 has been released. Regular homework 11 will be released soon. Both will be due next Wednesday (last day of classes).

2 Recap

Because of the long gap caused by the Thanksgiving break, I went back over the definitions of equivalence relation, partition, and basic properties such as transitive. See previous lecture notes.

3 Need for the RST properties

To build equivalence classes, our relation does need to be an equivalence relation, i.e. be reflexive, symmetric, and transitive. Suppose our relation wasn't reflexive. Then $[x]$ wouldn't necessarily contain x. In fact, it's possible that $[x]$ might not contain any elements and/or x might not be in any equivalence class.

Suppose our relation wasn't symmetric. Then y might be in $[x]$ but x not in $[y]$. So $[x]$ and $[y]$ could intersect partially, without being the same set, and it might matter which representative element we picked to name each equivalence class. This would make equivalence classes harder to use for mathematical or practical purposes.

Relations that aren't transitive fail in a more subtle way. Consider the relation R on the integers defined by xRy if and only if $|x - y| \leq 2$. Then $[3] = \{1, 2, 3, 4, 5\}$ and $[5] = \{3, 4, 5, 6, 7\}$. So, again, our subsets can intersect partially.

4 RST implies partition

Let's prove that if R is an equivalence relation on a set A, the equivalence classes of R form a partition of A.

First, since R is reflexive, xRx for any $x \in A$. So $x \in [x]$ for any $x \in A$. Therefore, no equivalence class is empty and the union of all equivalence classes is the whole set A. So the only thing that remains to be shown is that two distinct equivalence classes don't overlap.

Let x and y be two elements of A and suppose that $[x] \cap [y] \neq \emptyset$. We need to show that $[x] = [y]$.

Since $[x] \cap [y] \neq \emptyset$, we can pick an element c that is in $[x] \cap [y]$. I.e. $c \in [x]$ and $c \in [y]$. By the definition of equivalence class, this means that xRc and yRc. Since R is symmetric, we also have that cRy. So, by transitivity, xRy. And, thus by symmetry, yRx.

To show that $[x] \subseteq [y]$, pick any element $d \in [x]$. The definition of $[x]$ implies that xRd. Since we know that yRx, transitivity implies that yRd.

So $d \in [y]$. A similar argument shows that $[y] \subseteq [x]$. So $[x] = [y]$.

5 Proving that a relation is an equivalence relation

So far, I've assumed that we can just look at a relation and decide if it's an equivalence relation. Suppose someone asks you (e.g. on an exam) to prove that something is an equivalence relation. These proofs just use techniques you've seen before. Let's do a couple examples.

First, remember that we had defined the set of fractions F to contain all fractions $\frac{x}{y}$ where $x, y \in \mathbb{Z}$ and $y \neq 0$. We constructed the rational numbers from the fractions by defining certain pairs of fractions to be equivalent. Specifically, our equivalence relation $\sim$ was defined by: $\frac{x}{y} \sim \frac{p}{q}$ if and only if $xq = yp$.

Let's show that $\sim$ is an equivalence relation.

> Proof: Reflexive: For any x and y, $xy = xy$. So the definition of $\sim$ implies that $\frac{x}{y} \sim \frac{x}{y}$.
>
> Symmetric: if $\frac{x}{y} \sim \frac{p}{q}$ then $xq = yp$, so $yp = xq$, so $py = qx$, which implies that $\frac{p}{q} \sim \frac{x}{y}$.
>
> Transitive: Suppose that $\frac{x}{y} \sim \frac{p}{q}$ and $\frac{p}{q} \sim \frac{s}{t}$. By the definition of $\sim$, $xq = yp$ and $pt = qs$. So $xqt = ypt$ and $pty = qsy$. Since $ypt = pty$, this means that $xqt = qsy$. Cancelling out the q's, we get $xt = sy$. By the definition of $\sim$, this means that $\frac{x}{y} \sim \frac{s}{t}$.
>
> Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation.

Notice that the proof has three sub-parts, each showing one of the key properties. Each part involves using the definition of the relation, plus a small amount of basic math. The reflexive case is very short. The symmetric case is often short as well. Most of the work is in the transitive case.

There are relations for which one or the other property is hard to prove. Sometimes even reflexive can require some work. But it's more typical for these proofs to be almost mechanical.

6 Proving antisymmetry

Here's another example of a relation, this time an order (not an equivalence) relation. Consider the set of intervals on the real line $J = \{(a,b) \mid a,b \in \mathbb{R} \text{ and } a \leq b\}$. Define the containment relation C as follows:

$(a,b)\ C\ (c,d)$ if and only if $a \leq c$ and $d \leq b$

Let's show that C is antisymmetric.

Notice that C is a relation on intervals, i.e. pairs of numbers, not single numbers. If we substitute the definition of C into the definition of antisymmetric, what we need to show is that $(a,b)\ C\ (c,d)$ and $(c,d)\ C\ (a,b)$ implies that $(a,b) = (c,d)$.

So, suppose that we have two intervals (a,b) and (c,d) such that $(a,b)\ C\ (c,d)$ and $(c,d)\ C\ (a,b)$. By the definition of C, $(a,b)\ C\ (c,d)$ implies that $a \leq c$ and $d \leq b$. Similarly, $(c,d)\ C\ (a,b)$ implies that $c \leq a$ and $b \leq d$.

Since $a \leq c$ and $c \leq a$, $a = c$. Since $d \leq b$ and $b \leq d$, $b = d$. So $(a,b) = (c,d)$.

7 Proving that an operation is well-defined

Getting back to our rational number example, let's look at how to define operations on this new type of numbers. Given that we know how to do arithmetic on integers, we can define addition of fractions by

$$\frac{x}{y} + \frac{p}{q} = \frac{xq + py}{yq}$$

We'd like to apply this definition to our new rational numbers, by simply throwing equivalence class brackets around the inputs and outputs:

$$[\frac{x}{y}] + [\frac{p}{q}] = [\frac{xq+py}{yq}]$$

To work right, the output of this definition should not depend on which representative we pick for the input equivalence classes. E.g. $[\frac{2}{3}] + [\frac{10}{2}]$ should be equal to (say) $[\frac{-4}{-6}] + [\frac{5}{1}]$. In math jargon, we say that we want to make sure the operation is *well-defined* on the equivalence classes.

Suppose we defined a function $\oslash$ by $[\frac{x}{y}] \oslash [\frac{p}{q}] = [\frac{xy}{pq}]$. Then $[\frac{2}{3}] \oslash [\frac{10}{2}] = [\frac{6}{20}]$. Suppose we pick different representatives for the two input equivalence classes: $[\frac{-4}{-6}] \oslash [\frac{5}{1}] = [\frac{24}{5}]$. This produces a different output equivalence class because $\frac{6}{20} \not\sim \frac{24}{5}$. So this function isn't well-defined, a diplomatic way of saying that this is a buggy definition that doesn't work right.

Let's prove that our definition of rational addition is well-defined. Suppose that we pick two different representatives for each input: $\frac{x}{y} \sim \frac{v}{w}$ and $\frac{p}{q} \sim \frac{r}{s}$. Then we need to show that the two outputs we get are equivalent: $\frac{xq+py}{yq} \sim \frac{vs+wr}{ws}$

Since $\frac{x}{y} \sim \frac{v}{w}$, $xw = yv$. So $xwqs = yvqs$.

Since $\frac{p}{q} \sim \frac{r}{s}$, $ps = qr$. So $psyw = qryw$.

Adding these equations together: $xwqs + psyw = yvqs + qryw$. So $(xq + py)ws = (vs + wr)yq$. So, by the definition of $\sim$, $\frac{xq+py}{yq} \sim \frac{vs+wr}{ws}$.