Relations

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This lecture introduces relations and covers basic properties of relations, i.e. parts of section 8.1 and 8.3 of Rosen. When you look at Rosen, be aware that we're covering only relations on a single set, where he also covers relations between two sets.

1 Announcements

Over break, I plan to post the final homeworks: HW 11 and honors HW 4. Both will be due on the last day of classes. I plan to time their length on the assumption that no one will even glance at them until they get back from break.

There will be a quiz the Wednesday after break, covering material through the end of this week (lecture 36). Detailed skills list will appear over break.

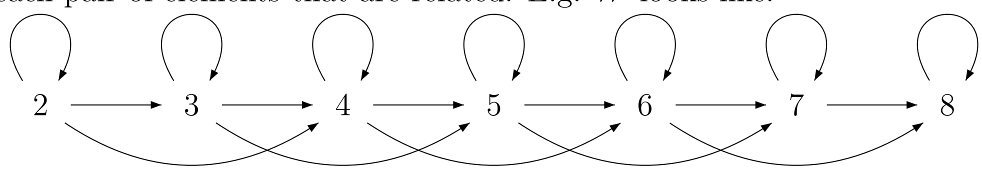
On the last Monday class, I plan to offer a makeup quiz for those folks who missed a quiz without an excuse (e.g. slept through it). The scores on this quiz will be discounted 20will cover all the material through quiz 3, in some unpredictable mixture.

2 Relations

A *relation R on a set A* is a subset of $A \times A$, i.e. R is a set of ordered pairs of elements from A. If R contains the pair (x, y), we say that x is related to

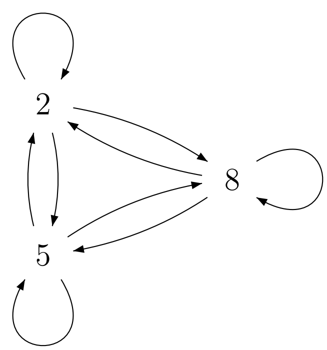
y, or xRy in shorthand.[1] We'll write $x \not R y$ to mean that x is not related to y.

For example, suppose we let $A = \{2, 3, 4, 5, 6, 7, 8\}$. We can define a relation W on A by xWy if and only if $x \leq y \leq x + 2$. Then W contains pairs like $(3, 4)$ and $(4, 6)$ and $(5, 5)$, but not the pairs $(6, 4)$ and $(3, 6)$. We can draw pictures of relations using directed graphs, with an arrow joining each pair of elements that are related. E.g. W looks like:



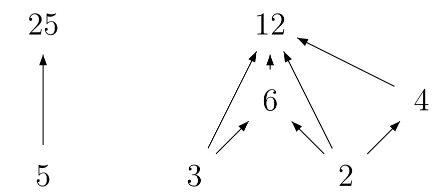
In fact, there's very little formal difference between a relation on a set A and a directed graph, because graph edges can be represented as ordered pairs of endpoints. They are two ways of describing the same situation.

We can define another relation S on A by saying that xSy is in S if $x \equiv y \pmod 3$. Then S would look like:

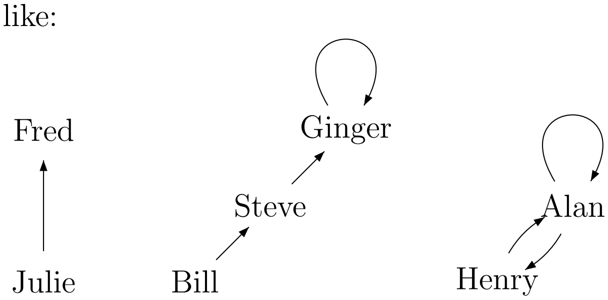


Or, suppose that $B = \{2, 3, 4, 5, 6, 12, 25\}$. Let's set up a relation T on B such that xTy if $x|y$ and $x \neq y$. Then our picture would look like

[1]The textbook writes $(x, y) \in R$ for the first few sections and then switches to this shorthand. I find the shorthand much easier to read.



Mathematical relations can also be used to represent real-world relationships, in which case they often have a less regular structure. For example, suppose that we have a set of students and student x is related to student y if x nominated y for ACM president. The graph of this relation (call it Q) might look like:



Relations can also be defined on infinite sets or multi-dimensional objects. For example, we can define a relation Z on the real plane $\mathbb{R}^2$ in which (x, y) is related to (p, q) if and only if $x^2 + y^2 = p^2 + q^2$. In other words, two points are related if they are the same distance from the origin.

For complex relations, the full directed graph picture can get a bit messy. So there are simplified types of diagrams for certain specific special types of relations, e.g. the so-called Hasse diagram for partial orders.

3 Properties of relations: reflexive

Relations are classified by several key properties. The first is whether an element of the set is related to itself or not. There are three cases

- Reflexive: every element is related to itself.
- Irreflexive: no element is related to itself.
- Neither reflexive nor irreflexive: some elements are related to themselves but some aren't.

In our pictures above, elements related to themselves have self-loops. So it's easy to see that W and S are reflexive, T is irreflexive, and Q is neither. The familiar relations $\leq$ and $=$ on the real numbers are reflexive, but $<$ is irreflexive. Suppose we define a relation M on the integers by xMy if and only if $x + y = 0$. Then 2 isn't related to itself, but 0 is.

The formal definition states that if R is a relation on a set A then

- R is reflexive if xRx for all $x \in A$.
- R is irreflexive if $x \not R x$ for all $x \in A$.

Notice that irreflexive is not the negation of reflexive. The negation of reflexive would be:

- not reflexive: there is an $x \in A, x \not R y$

4 Symmetric and antisymmetric

Another important property of a relation is whether the order matters within each pair. That is, if (x, y) is in R, is (y, x) always in R? A relation satisfying this property is called *symmetric*. In a graph picture of a symmetric relation, a pair of elements is either joined by a pair of arrows going in opposite directions, or no arrows. In our examples with pictures above, only S is symmetric.

Relations that resemble equality are normally symmetric. For example, the relation X on the integers defined by xXy iff $|x| = |y|$ is symmetric. So is the relation N on the real plane defined by $(x, y)N(p, q))$ iff $(x-p)^2+(y-q)^2 \leq 25$ (i.e. the two points are no more than 5 units apart).

Relations that put elements into an order, like $\leq$ or divides, have a different property called *antisymmetry*. A relation is *antisymmetric* if two distinct elements are never related in both directions. In a graph picture, a pair of points may be joined by a single arrow, or not joined at all. In our pictures above, W and T are antisymmetric.

As with reflexivity, there are mixed relations that have neither property. So the relation Q above is neither symmetric nor antisymmetric.

If R is a relation on a set A, here's the formal definition of what it means for R to be symmetric (which doesn't contain anything particularly difficult):

symmetric: for all $x, y \in A, xRy$ implies yRx

There's two ways to define antisymmetric. They are logically equivalent and you can pick whichever is more convenient for your purposes:

antisymmetric: for all x and y in A with $x \neq y$, xRy implies $y \not R x$

antisymmetric: for all x and y in A, xRy and yRx implies $x = y$

To interpret the second definition, remember that when mathematicians pick two values x and y, they leave open the possibility that the two values are actually the same. If we said that in normal conversational English, we would normally mean that they had to be different. I find that the first definition is better for understanding the idea of antisymmetry, but the second is more useful for writing proofs.

5 Transitive

The final important property of relations is transitivity. A relation R on a set A is *transitive* if

transitive: for all $a, b, c \in A, aRb$ and bRc implies that aRc

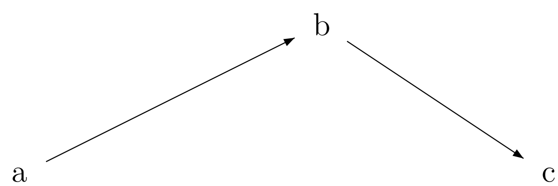
You've probably seen transitivity before, because it holds for a broad range of familiar numerical relations such as $<$, $=$, divides, and set inclusion. For example, for real numbers, if $x < y$ and $y < z$, then $x < z$. Similarly, if $x|y$ and $y|z$, then $x|z$. For sets, $X \subseteq Y$ and $Y \subseteq Z$ implies that $X \subseteq Z$.

If we look at graph pictures, transitivity means that whenever there is a path from x to y then there must be a direct arrow from x to y. This is true for S and B above, but not for W or Q.

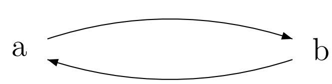
We can also understand this by spelling out what it means for a relation R on a set A not to be transitive:

$$\text{not transitive: there are } a, b, c \in A, aRb \text{ and } bRc \text{ and } a \not R c$$

So, to show that a relation is not transitive, we need to find one counter-example, i.e. specific elements a, b, and c such that aRb and bRc but not aRc. In the graph of a non-transitive relation, you can find a subsection that looks like:

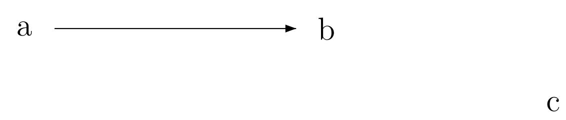


It could be that a and c are actually the same element, in which case the offending subgraph might look like:



The problem here is that if aRb and bRa, then transitivity would imply that aRa and bRb.

One subtle point about transitive is that it's ok if some sets of elements just aren't connected at all. For example, this subgraph is consistent with the relation being transitive.



A disgustingly counter-intuitive special case is the relation $P = \emptyset$ on any non-empty set, i.e. the relation in which no elements are related to one another. It's transitive, because it's never possible to satisfy the hypothesis of the definition of transitive. It's also symmetric, for the same reason. And, oddly enough, antisymmetric.

This special-case relation is irreflexive and not reflexive. Unlike symmetry and transitivity, reflexivity unconditionally requires that pairs of the form (x, x) must be in the relation. In this case, they aren't.

6 Types of relations

Now that we have these basic properties defined, we can define three important classes of relations:

- An equivalence relation is a relation that is reflexive, symmetric, and transitive.
- A partial order is a relation that is reflexive, antisymmetric, and transitive.
- A strict partial order is a relation that is irreflexive, antisymmetric, and transitive.

Equivalence relations act like equality, partial orders act like $\leq$ or $\geq$, and strict partial orders act like $<$ or $>$. In the picture examples above, S is an equivalence relation and T is a strict partial order. If we pick some collection of sets, then the set inclusion relation $\subseteq$ (no picture) would be a (non-strict) partial order on them.