Graphs II

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This lecture continues the discussion of graphs (section 9.2 of Rosen).

The lecture was delivered by Lucas and transcribed into latex by Margaret.

1 Announcements

We still don't know what will happen with the possible TA strike next week and we'll just have to play it by ear.

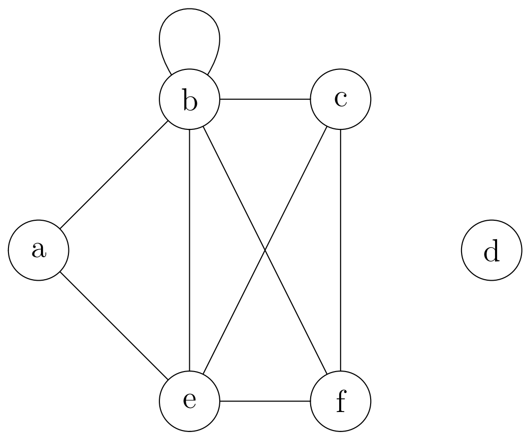
2 Recap

Recall that a graph is a pair (V, E), where V is a set of vertices (also known as nodes) and E is a set of edges. For a directed graph, each edge is an ordered pair of vertices, i.e. $E \in V^2$. For an undirected graph, we use unordered pairs of vertices, which can be written using set notation e.g. $\{v_1, v_2\}$.

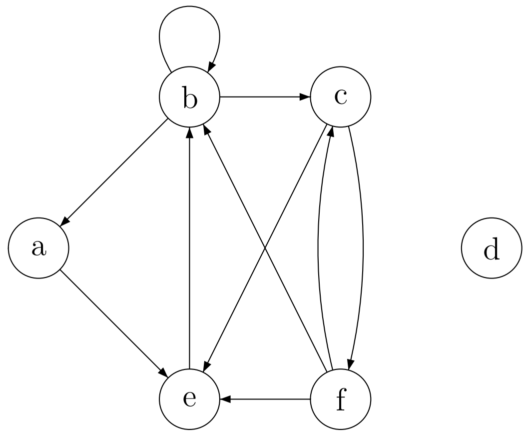
An undirected graph has "2-way" edges, which can be traversed in both directions. For example, most streets or facebook friend relationships (which must be accepted by the other person). A directed graph has one-directional edges. For example, twitter following is a directed relationship. A celebrity may be followed by many people, few of whom the celebrity follows.

3 Degrees

In an undirected graph, the degree of a vertex v, written $deg(v)$ is the number of edges incident to v (i.e. having v as an endpoint). Self-loops, if you are allowing them, count twice. For example, in the following graph, a has degree 2, b has degree 6, d has degree 0, and so forth.



In a directed graph, each vertex has an in-degree (written $deg^{-}(v)$) which is the number of incoming edges, and an out-degree (written $deg^{+}(v)$) which is the number of outgoing edges. The degree $deg(v)$ is then the sum of the in-degree and the outdegree of v. For example, in the following graph, f has in-degree 1 and out-degree 3. The previous graph is called the "underlying undirected graph" for this one, i.e. it's what you get if you remove the directionality from each edge (and merge pairs of edges joining the same two nodes in different directions).



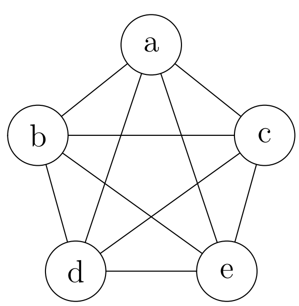
In an undirected graph, each edge contributes to two vertex degrees. So the sum of the degrees of all the vertices is twice the number of edges. This is called the Handshaking Theorem and can be written as

$$\sum_{v \in V} deg(v) = 2|E|$$

4 Complete graphs

Recall that a simple graph is undirected, has no self-loops and no multi-edges. We'll see five special types of graphs, all of which are simple graphs.

The complete graph on n vertices ($n \geq 1$) shorthand name K_n, is a graph with n vertices in which every vertex is connected to every other vertex. K_5 is shown below. In principle, you could have a graph with no vertices, so there could be a K_0, but we won't worry about that this term.



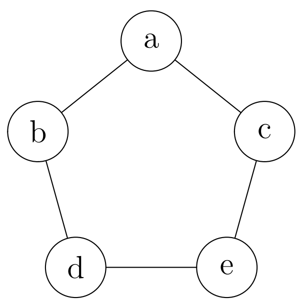
To calculate the number of edges in K_n, think about the situation from the perspective of the first vertex. It is connected to $n-1$ other vertices. If we look at the second vertex, it has $n-2$ more connections. And so forth. So we have $\sum_{k=1}^{n} n-k = \frac{n(n-1)}{2}$ edges. Another way to do this calculation is to notice that there are $\binom{n}{2}$ ways to pick a subset of two vertices. $\binom{n}{2}$ is also equal to $\frac{n(n-1)}{2}$.

5 Cycles and wheels

Suppose that we have n vertices named $v_1, \ldots, v_n$. Then the The cycle C_n $(n \geq 3)$ is the graph with these vertices and edges

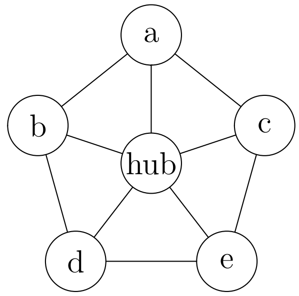
$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$$

So C_5 looks like



C_n has n vertices and also n edges. Cycle graphs often occur in networking applications. They could also be used to model games like "telephone" where people sit in a circle and communicate only with their neighbors.

The wheel W_n is just like the cycle C_n except that it has an additional central "hub" node which is connected to all the others. Notice that W_n has $n+1$ nodes (not n nodes). It has $2n$ edges. For example, W_5 looks like

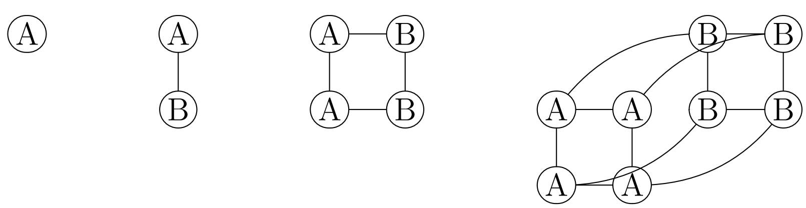


6 Hypercubes

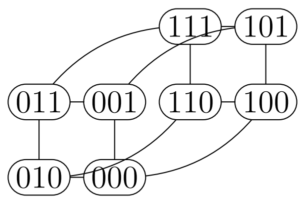
An n-cube or a hypercube Q_n is the graph of the corners and edges of an n-dimensional cube. It defines a binary coordinate space for dimension n. It is defined recursively as follows (for any $n \in \mathbb{N}$):

1. Q_0 is a single vertex with no edges
2. Q_n consists of two copies of Q_{n-1} with edges joining corresponding vertices.

That is, each node v_i in one copy of Q_{n-1} is joined by an edge to its clone copy v_i' in the second copy of Q_{n-1}. Q_0, Q_1, Q_2, and Q_3 look as follows. The node labels distinguish the two copies of $Qn - 1$



To build a binary coordinate system, we label nodes with binary numbers, where each binary digit corresponds to the value of one coordinate.



Q^n has 2^n nodes. To compute the number of edges, we set up a recurrence:

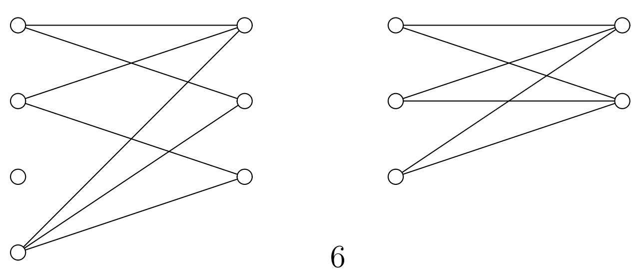
1. $E(0) = 0$
2. $E(n) = 2E(n-1) + 2^{n-1}$

The 2^{n-1} term is the number of nodes in each copy of Q^{n-1}, i.e. the number of edges required to join corresponding nodes.

7 Bipartite graphs

The last special type of graph is a bipartite graph. A graph $G = (V, E)$ is bipartite if we can split V into two non-overlapping subsets V_1 and V_2 such that every edge in G connects an element of V_1 with an element of V_2. That is, no edge connects two nodes from the same part of the division. Bipartite graphs often appear in matching problems, where the two subsets represent different types of objects, e.g. matching a group of women with a group of male study partners.

The complete bipartite graph $K_{m,n}$ is a bipartite graph with m nodes in V_1, m nodes in V_2, and which contains all possible edges that are consistent with the definition of bipartite. The diagram below shows a partial bipartite graph on a set of 7 nodes, as well as the complete bipartite graph $K3, 2$.



The complete bipartite graph $K_{m,n}$ has $m + n$ nodes and mn edges.