Induction II

Margaret M. Fleck

5 October 2009

In this lecture, we see more examples of mathematical induction (section 4.1 of Rosen).

1 Recap

If we want to prove a claim $P(n)$, for all integers n, a proof by induction has the following outline:

> Proof: We will show $P(n)$ is true for all n, using induction on n.
>
> **Base**: We need to show that $P(1)$ is true.
>
> **Induction**: Suppose that $P(k)$ is true, for some integer k. We need to show that $P(k+1)$ is true.

2 Why is this legit?

There are several ways to think about mathematical induction, and understand why it's a legitimate proof technique. Different people prefer different motivations at this point, so I'll offer several.

A proof by induction of that $P(k)$ is true for all positive integers k involves showing that $P(1)$ is true (base case) and that $P(k) \rightarrow P(k+1)$ (inductive step).

Domino Theory: Imagine an infinite line of dominoes. The base step pushes the first one over. The inductive step claims that one domino falling down will push over the next domino in the line. So dominos will start to fall from the beginning all the way down the line. This process continues forever, because the line is infinitely long. However, if you focus on any specific domino, it falls after some specific finite delay.

Recursion fairy: The recursion fairy is the mathematician's version of a programming assistant. Suppose you tell her how to do the proof for $P(1)$ and also why $P(k)$ implies $P(k+1)$. Then suppose you pick any integer (e.g. 1034) then she can take this recipe and use it to fill in all the details of a normal direct proof that P holds for this particular integer. That is, she takes $P(1)$, then uses the inductive step to get from $P(1)$ to $P(2)$, and so on up to $P(1034)$.

Smallest counter-example: Let's assume we've established that $P(1)$ is true and also that $P(k)$ implies $P(k+1)$. Let's prove that $P(j)$ is true for all positive integers j, by contradiction.

That is, we suppose that $P(1)$ is true, also that $P(k)$ implies $P(k+1)$, but there is a counter-example to our claim that $P(j)$ is true for all j. That is, suppose that $P(m)$ was not true for some integer m.

Now, let's look at the set of all counter-examples. We know that all the counter-examples are larger than 1, because our induction proof established explicitly that $P(1)$ was true. Suppose that the smallest counter-example is s. So $P(s)$ is true. We know that $s > 1$, since $P(1)$ was true. Since s was supposed to be the smallest counter-example, $s-1$ must not be a counter-example, i.e. $P(s-1)$ is true.

But now we know that $P(s-1)$ is true but $P(s)$ is not true. This directly contradicts our assumption that $P(k)$ implies $P(k+1)$ for any k.

The smallest counter-example explanation is a formal proof that induction works, given how we've defined the integers. If you dig into the mathematics, you'll find that it depends on the integers having what's called the "well-ordering" property: any subset has a smallest element. Standard axioms used to define the integers include either a well-ordering or an induction axiom.

These arguments don't depend on whether our starting point is 1 or some other integer, e.g. 0 or 2 or -47. All you need to do is ensure that your base case covers the first integer for which the claim is supposed to be true.

3 Building an inductive proof

In constructing an inductive proof, you've got two tasks. First, you need to set up this outline for your problem. This includes identifying a suitable proposition P and a suitable integer variable n.

Notice that $P(n)$ must be a statement, i.e. something that is either true or false. For example, it is **never** just a formula whose value is a number. Also, notice that $P(n)$ must depend on an integer n. This integer n is known as our *induction variable*. The assumption at the start of the inductive step ("$P(k)$ is true") is called the inductive hypothesis.

Your second task is to fill in the middle part of the induction step. That is, you must figure out how to relate a solution for a larger problem $P(k+1)$ to a solution for a small problem $P(k)$. Most students want to do this by starting with the small problem and adding something to it. For more complex situations, it's usually better to start with the larger problem and try to find an instance of the smaller problem inside it.

4 Finishing up our example

Last Friday, we used induction to prove the following claim:

Claim 1 *For any positive integer n, $\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$.*

Let's do another example:

Claim 2 *For every positive integer $n \geq 4$, $2^n < n!$.*

Remember that $n!$ ("n factorial") is $1 \cdot 2 \cdot 3 \cdot 4 \ldots n$. E.g. $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

First, as usual, try some specific integers and verify that the claim is true. Since the claim specifies $n \geq 4$, it's worth checking that 4 does work but the smaller integers don't.

In this claim, the proposition $P(n)$ is $2^n < n!$. So an outline of our inductive proof looks like:

> Proof: Suppose that n is an integer and $n \geq 4$. We'll prove that $2^n < n!$ using induction on n.
>
> Base: $n = 4$. [show that the formula works for $n = 4$]
>
> Induction: Suppose that the claim holds for $n = k$. That is, suppose that we have an integer $k \geq 4$ such that $2^k < k!$. We need to show that the claim holds for $n = k+1$, i.e. that $2^{k+1} < (k+1)!$.

Notice that our base case is for $n = 4$ because the claim was specified to hold only for integers ≥ 4.

Fleshing out the details of the algebra, we get the following full proof. When working with inequalities, it's especially important to write down your assumptions and what you want to conclude with. You can then work from both ends to fill in the gap in the middle of the proof.

> Proof: Suppose that n is an integer and $n \geq 4$. We'll prove that $2^n < n!$ using induction on n.
>
> Base: $n = 4$. In this case $2^n = 2^4 = 16$. Also $n! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. Since $16 < 24$, the formula holds for $n = 4$.
>
> Induction: Suppose that the claim holds for $n = k$. That is, suppose that we have an integer $k \geq 4$ such that $2^k < k!$. We need to show that $2^{k+1} < (k+1)!$.
>
> $2^{k+1} = 2 \cdot 2^k$. By the inductive hypothesis, $2^k < k!$, so $2 \cdot 2^k < 2 \cdot k!$. Since $k \geq 4$, $2 < k+1$. So $2 \cdot k! < (k+1) \cdot k! = (k+1)!$.
>
> Putting these equations together, we find that $2^{k+1} < (k+1)!$, which is what we needed to show.

5 Some comments about style

Notice that the start of the proof tells you which variable in your formula (n in this case) is the induction variable. In this formula, the choice of induction variable is fairly obvious. But sometimes there's more than one integer floating around that might make a plausible choice for the induction variable. It's good style to always mention that you are doing a proof by induction and say what your induction variable is.

It's also good style to label your base and inductive steps.

Notice that the proof of the base case is very short. In fact, I've written about about twice as long as you'd normally see it. Almost all the time, the base case is trivial to prove and fairly obvious to both you and your reader. Often this step contains only some worked algebra and a check mark at the end. However, it's critical that you do check the base case. And, if your base case involves an equation, compute the results for both sides (not just one side) so you can verify they are equal.

The important part of the inductive step is ensuring that you assume $P(k)$ and use it to show $P(k+1)$. At the start, you must spell out your inductive hypothesis, i.e. what $P(k)$ is for your claim. Make sure that you use this information in your argument that $P(k+1)$ holds. If you don't, it's not an inductive proof and it's very likely that your proof is buggy.

At the start of the inductive step, it's also a good idea to say what you need to show, i.e. quote what $P(k+1)$ is.

These "style" issues are optional in theory, but actually critical for beginners writing inductive proofs. You will lose points if your proof isn't clear and easy to read. Following these style points (e.g. labelling your base and inductive steps) is a good way to ensure that it is, and that the logic of your proof is correct.

6 Another example

The previous examples applied induction to an algebraic formula. We can also apply induction to other sorts of statements, as long as they involve a suitable integer n.

Claim 3 *For any natural number n, $n^3 - n$ is divisible by 3.*

In this case, $P(n)$ is "$n^3 - n$ is divisible by 3."

Proof: By induction on n.

Base: Let $n = 0$. Then $n^3 - n = 0^3 - 0 = 0$ which is divisible by 3.

Induction: Suppose that $k^3 - k$ is divisible by 3, for some positive integer k. We need to show that $(k+1)^3 - (k+1)$ is divisible by 3.

$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = (k^3 - k) + 3(k^2 + k)$

From the inductive hypothesis, $(k^3 - k)$ is divisible by 3. And $3(k^2 + k)$ is divisible by 3 since $(k^2 + k)$ is an integer. So their sum is divisible by 3. That is $(k+1)^3 - (k+1)$ is divisible by 3. □

The zero base case is technically enough to make the proof solid, but sometimes a zero base case doesn't provide good intuition or confidence. So you'll sometimes see an extra base case written out, e.g. $n = 1$ in this example, to help the author or reader see why the claim is plausible.

7 Induction on the size of sets

Now, let's consider a fact about sets which we've used already but never properly proved:

Claim 4 *For any finite set S containing n elements, S has 2^n subsets.*

The objects involved in this claim are sets. To apply induction to facts that aren't about the integers, we need to find a way to use the integers to organize our objects. In this case, we'll organize our sets by their cardinality.

The proposition $P(n)$ for our induction is then "For any set S containing n elements, S has 2^n subsets." Notice that each $P(k)$ is a claim about a whole family of sets, e.g. $P(1)$ is a claim about $\{37\}$, $\{\text{fred}\}$, $\{-31.7\}$, and so forth.

> Proof: We'll prove this for all sets S, by induction on the cardinality of the set.
>
> Base: Suppose that S is a set that contain no elements. Then S is the empty set, which has one subset, i.e. itself. Putting zero into our formula, we get $2^0 = 1$ which is correct.
>
> Induction: Suppose that our claim is true for all sets of k elements, where k is some non-negative integer. We need to show that it is true for all sets of $k+1$ elements.
>
> Suppose that S is a set containing $k+1$ elements. Since k is non-negative, $k+1 \geq 1$, so S must contain at least one element. Let's pick a random element a in S. Let $T = S - \{a\}$.
>
> If B is a subset of S, either B contains a or B doesn't contain a. The subsets of S not containing a are exactly the subsets of T. The subsets of S containing a are exactly the subsets of T, with a added to each one. So S has twice as many subsets as T.
>
> By the induction hypothesis, T has 2^k subsets. So S has $2 \cdot 2^k = 2^{k+1}$ subsets, which is what we needed to show.

Notice that, in the inductive step, we need to show that our claim is true for **all** sets of $k+1$ elements. Because we are proving a universal statement, we need to pick a representative element of the right type. This is the set S that we choose in the second paragraph of the inductive step.