Sets II

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This lecture shows how to prove facts about set equality and set inclusion. (Rosen section 2.2).

1 Announcements

First midterm is a week from Wednesday (i.e. September 30th) in class. Please tell me ASAP about any conflicts.

Quizzes will be handed out in sections today and tomorrow. (Grades should be on compass this afternoon.)

2 Recap

Last class, we saw a bunch of set theory notation and operations on sets. Much of this was (more or less) familiar. The least familiar operations were

- Powerset: $\mathbb{P}(A) = \{\text{all subsets of } A\}$
- Cartesian product: $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

Remember that (x, y) is an ordered pair containing two objects, which is very different from the two-object set $\{x, y\}$. In (x, y), the order of the two elements matters and you can have duplicates e.g. $(2, 2)$. If you add the same thing to a set twice, you only get one copy in the set, e.g. $\{2, 2\} = \{2\}$.

3 Set identities

Rosen lists a large number of identities showing when two sequences of set operations yield the same output sets. For example:

DeMorgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

I won't go through these in detail because they are largely identical to the identities you saw for logical operations, if you make the following correspondences:

- $\cup$ is like $\vee$
- $\cap$ is like $\wedge$
- $\overline{A}$ is like $\neg P$
- $\emptyset$ (the empty set) is like F
- U (the universal set) is like T

The two systems aren't exactly the same. E.g. set theory doesn't use a close analog of the $\rightarrow$ operator. But they are very similar.

You can use these set theory identities to prove new identites via a chain of equalities. This is exactly like what you did with logical identities and there's no point in walking you through the same exercise twice.

I'm also going to skip showing you set membership tables, which are in Rosen, because those are just like truth tables and I'm confident you're already on top of that idea.

4 Proving facts about set inclusion

So far in school, most of your proofs or derivations have involved reasoning about equality. Inequalities (e.g. involving numbers) have been much less commmon. With sets, the situation is reversed. Proofs typically involve reasoning about subset relations, even when proving two sets to be equal. Proofs that rely primarily on a chain of set equalities do occur, but they are much less common. Even when both approaches are possible, the approach based on subset relations is often easier to write and debug.

As a first example of a typical set proof, consider the following claim:

Claim 1 *For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

This property is called "transitivity," just like similar properties for (say) $\leq$ on the real numbers. Both $\subseteq$ and $\leq$ are examples of a general type of object called a *partial order*, for which transitivity is a key defining property.

First, remember our definition of $\subseteq$: a set A is a subset of a set B if and only if, for any object x, $x \in A$ implies that $x \in B$.

> Proof: Let A, B, and C be sets and suppose that if $A \subseteq B$ and $B \subseteq C$.

Our ultimate goal is to show that $A \subseteq C$. This is an if/then statement: for any x, if $x \in A$, then $x \in C$. So we need to pick a representative x and assume the hypothesis is true, then show the conclusion. So our proof continues:

> Let x be an element of A. Since $A \subseteq B$ and $x \in A$, then $x \in B$ (definition of subset). Similarly, since $x \in B$ and $B \subseteq C$, $x \in C$. So for any x, if $x \in A$, then $x \in C$. So $A \subseteq C$ (definition of subset again). □

5 Example proof: deMorgan's law

Two strategies are commonly used for proving that two sets A and B are equal. One method is to show the equality via a chain of equalities. This is great when it works but it requires each step to work in both directions.

The more common strategy is to show that $A \subseteq B$ and $B \subseteq A$, using separate subproofs. We can then conclude that $A = B$. This is just like showing that $x = y$ by showing that $x \leq y$ and $y \leq x$. Although it seems like more trouble to you right now, this is a more flexible approach that works in a wider range of situations, especially in upper-level computer science and mathematics courses (e.g. real analysis, algorithms).

As an example, let's look at

Claim (DeMorgan's Law): For any sets A and B, $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Let's prove this in both styles:

Proof 1: Let A and B be sets. Then

$$\begin{aligned}
\overline{A \cup B} &= \{x \mid x \notin A \cup B\} \\
&= \{x \mid \text{ it's not the case that } (x \in A \text{ or } x \in B)\} \\
&= \{x \mid x \notin A \text{ and } x \notin B\} \\
&= \{x \mid x \in \overline{A} \text{ and } x \in \overline{B}\} \\
&= \overline{A} \cap \overline{B}
\end{aligned}$$

Proof 2: Let A and B be sets. We'll do this in two parts:

$\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$: Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$. So it's not the case that ($x \in A$ or $x \in B$. So (by deMorgan's Law for logic), $x \notin A$ and $x \notin B$. That is $x \in \overline{A}$ and $x \in \overline{B}$. So $x \in \overline{A} \cap \overline{B}$.

$\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$: very similar.

In this example, the second half is very basically the first half written backwards. However, this is not always the case. The power of this method comes from the fact that you can apply different techniques to proving the two subset relations.

6 An example with products

Let's move on to some facts that aren't so obvious, because they involve Cartesian products. (Next lecture, we'll see some facts involving power sets.)

Claim 2 *If A, B, C, and D are sets such that $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.*

To prove this, we first gather up all the facts we are given. What we need to show is the subset inclusion $A \times C \subseteq B \times D$. To do this, we'll need to pick a representative element from $A \times C$ and show that it's an element of $B \times D$.

> Proof: Let A, B, C, and D be sets and suppose that $A \subseteq B$ and $C \subseteq D$.
>
> Let $p \in A \times C$. By the definition of Cartesian product, p must have the form (x, y) where $x \in A$ and $y \in C$.
>
> Since $A \subseteq B$ and $x \in A$, $x \in B$. Similarly, since $C \subseteq D$ and $y \in C$, $y \in D$. So then $p = (x, y)$ must be an element of $B \times D$.
>
> We've shown that, for all p, if $p \in A \times C$ then $p \in B \times D$. This means that $A \times C \subseteq B \times D$. □

The last paragraph is optional. When you first start, it's a useful recap because you might be a bit fuzzy about what you needed to prove. As you get experienced with this sort of proof, it's often omitted. But you will still see it occasionally at the end of a very long (e.g. multi-page) proof, where even an experienced reader might have forgotten the main goal of the proof.

7 Another example with products

Here's another claim about Cartesian products:

Claim 3 *For any sets A, B, and C, if $A \times B \subseteq A \times C$ and $A \neq \emptyset$, then $B \subseteq C$.*

Notice that this is like dividing both sides of an algebraic equation by a non-zero number: if $xy \leq xz$ and $x \neq 0$ then $y \leq z$. The claim fails if we allow x to be zero. Since the empty set plays the role of zero in set theory, this suggests why we have the analogous condition in our claim about sets. Although there are occasionally differences between sets and numbers, the parallelism is strong enough to suggest special cases that you should be sure to investigate.

A general property of proofs is that the proof should use all the information in the hypothesis of the claim. If that's not the case, either the proof has a bug (e.g. on a CS 173 homework) or the claim could be revised to make it more interesting (e.g. when doing a research problem, or a buggy homework problem). Either way, there's an important issue to deal with. So, in this case, we need to make sure that our proof does use the fact that $A \neq \emptyset$.

Here's a draft proof:

> Proof draft 1: Suppose that A, B, C, and D are sets and suppose that $A \times B \subseteq A \times C$ and $A \neq \emptyset$. We need to show that $B \subseteq C$.
>
> So let's choose some $x \in B$. ...

The main fact we've been given is that $A \times B \subseteq A \times C$. To use it, we need an element of $A \times B$. Right now, we only have an element of B. We need to find an element of A to pair it with. To do this, we reach blindly into A, pull out some random element, and give it a name. But we have to be careful here: what if A does contain any elements? So we have to use the assumption that $A \neq \emptyset$.

> Proof draft 1: Suppose that A, B, C, and D are sets and suppose that $A \times B \subseteq A \times C$ and $A \neq \emptyset$. We need to show that $B \subseteq C$.
>
> So let's choose some $x \in B$. Since $A \neq \emptyset$, we can choose an element t from A. Then $(t, x) \in A \times B$ by the definition of Cartesian product.
>
> Since $(t, x) \in A \times B$ and $A \times B \subseteq A \times C$, we must have that $(t, x) \in A \times C$ (by the definition of subset). But then (again by the definition of Cartesian product) $x \in C$.
>
> So we've shown that if $x \in B$, then $x \in C$. So $B \subseteq C$, which is what we needed to show.