

CS 173: Discrete Mathematical Structures, Fall 2009

Honors Homework 2

Due by 5pm on Wednesday 14 October. Please give to Margaret or push it under the door of Margaret's office (3214 Siebel).

Your homework must be formatted using latex. Please turn in hardcopy (latex output only, don't worry about the source).

A binary operation on a set A is a function that takes two elements of A and returns an element of A . For example, addition and multiplication are binary operations on the set of real numbers.

Suppose that $*$ is a binary operation on a set A . Then $*$ is *commutative* if

$$\forall x, y \in A, x * y = y * x$$

and $*$ is *associative* if

$$\forall x, y, z \in A, x * (y * z) = (x * y) * z$$

For example, multiplication of real numbers is commutative and associative, but subtraction is neither. E.g. $(3 - 2) - 1 = 0 \neq 2 = (3 - (2 - 1))$. However, the two properties are actually independent of one another.

1. Show that exponentiation (i.e. x^y) is neither commutative nor associative. Show that both properties fail using concrete counter-examples.
2. Let's define the operation Δ on the real numbers as follows:

$$x \Delta y = \frac{2xy}{x + y}$$

This is often called the "harmonic mean". Is Δ commutative? associative? Prove that your answers are correct. (Use a general argument if it's true, a concrete counter-example if it's false.)

3. Suppose that we have a three-letter sequence such as ABC . A permutation is a way to reorder the sequence. For example, one permutation k swaps the first and last characters. So $k(ABC) = CBA$ and $k(BCA) = ACB$. Another permutation (call it p) rotates the letters rightwards. So $p(ABC) = CAB$ and $p(CBA) = ACB$. Or, we can define a third permutation m which swaps the first two letters in the sequence, i.e. $m(ABC) = BAC$ and $m(CBA) = BCA$.

We can “multiply” two permutations by doing one after another. So $kp(ABC) = k(CAB) = BAC = m(ABC)$. So $kp = m$. (Notice the rightmost permutation in the product is applied first, so the order is like function composition.)

- (a) For a fixed input sequence of three letters, there are six possible outputs and thus six possible permutations. We’ve seen three above. Write down the other three. If you have trouble finding all three missing ones, it may help to start trying to build the table in part (b) and see if it suggests what might be missing.
- (b) Write out a multiplication table for the set of six permutations. That is, for each pair of input permutations, show which output permutation you get when you multiply them together.
- (c) Use a concrete counter-example to show that this multiplication operation is not commutative.