CS 173: Discrete Structures, Fall 2009
Homework 9 Solutions

This homework contains 4 problems worth a total of 50 points.

1. **Counting [18 points]**

   Solve the following counting problems, providing brief explanations and/or work to justify your answers. You do not need to multiply out factorials or trying to simplify formulas to reach your final answer. For example, $\frac{10!}{6!}$ would be fine as an answer.

   (a) Central Plains Pizza offers three choices for the diameter, two choices for crust thickness, and 12 choices of topping. Suppose that you limit yourself to 5 or fewer toppings. How many different pizzas can you construct?
   [Solution]
   There are $\sum_{i=0}^{5} \binom{12}{i}$ ways to choose toppings for a given diameter and crust. Thus:

   $$3 \cdot 2 \cdot \left(\sum_{i=0}^{5} \binom{12}{i} \right)$$

   (b) How many distinct strings can you make out of the characters in the word "potato-soup"? (Include the hyphen as a character.)
   [Solution]
   There are 11 characters, but we need to be careful about indistinguishable characters: 'p' occurs two times, 'o' occurs three times, and 't' occurs two times. We saw a formula in lecture for permutations with identical objects:

   $$\frac{11!}{2!3!2!}$$

   (c) You need to form a battle group of 11 made up of orcs, elves, and goblins. In how many ways can you choose the composition of your battle group?
   [Solution]
   This is a situation of combinations with repetition, with 3 types and 11 slots. The formula from lecture gives us:

   $$\binom{11+3-1}{3-1} = \frac{13!}{11!2!}$$

   Intuitively, we are choosing a place for $(3-1)$ dividers from $11 + (3-1)$ possible positions. For example:

   $$\ldots | \ldots | \ldots\ldots$$

   where the left dots are orcs, the middle dots are elves, the right dots are goblins, and the bars are the dividers. Note that the dividers can be adjacent: $\ldots || \ldots\ldots\ldots$

(d) You need to form a battle group consisting of an orc, an elf, and a goblin whose total strength is 12. The strength of each creature is an integer between 1 and 10 and the strength of the group is the sum of the individual strengths. In how many ways can you construct your battle group?

[Solution]

We can translate this into a string-like problem. There are 12 strength "items" to fill, and 3 member "types" to fill them (orc, elf, goblin). This is different from the previous problem, since there is a minimum bound on the strength and we cannot allow dividers to be adjacent.

We'll place the dividers between the 12 items so that the dividers will not be adjacent or at the edge. There are 11 possible divider positions amid 12 items, and we just need to choose 2 of them to separate 3 types:

$$\binom{11}{2} = \frac{11!}{9!2!}$$

Note that because there are only 12 strength items to fill, our solution will not accidentally exceed 10 strength for any member. The problem would have been more complicated if there were more than 12 items.

(e) You are out in the middle of the cornfields, where the (dirt) roads all run north-south or east-west, evenly spaced at one road per mile. If you start at a road intersection and wish to get to the intersection 5 miles to the north and 7 miles east, how many different routes could you take?

[Solution]

Every path of minimum length is 12 "steps", where each step is one mile to the north or east. To count the number of paths, we'll determine the order in which the 5 north steps and the 7 east steps are taken. We can find this by choosing the position of the 5 north steps in the 12 steps that will occur; the remaining steps will be the east ones. (Equivalently, we can start by choosing the 7 east steps from the 12.)

$$\binom{12}{5} = \binom{12}{7} = \frac{12!}{5!7!}$$

(f) Your latest cheapo cell phone keyboard only includes the uppercase alphabet (26 characters total). How many 12-character strings can you type that start with ST and contain no more than three T's?

[Solution]

After ST is fixed, there are 10 characters remaining to choose, and at most 2 of them are T's. There are $\sum_{i=0}^{2} \binom{10}{i}$ ways to place 2 or fewer T's in 10 positions. After the T's are placed, there are 25 possible letters (the alphabet minus T) for each of the remaining positions. Thus:

$$\sum_{i=0}^{2} \binom{10}{i} \cdot 25^{10-i}$$

2. **Combinatorial formula [8 points]**

   Use induction and Pascal's identity to prove the following formula holds for any positive integers s and n:

$$\sum_{k=0}^{n} \binom{s+k-1}{k} = \binom{s+n}{n}$$

   [**Solution**] We will prove this by induction on n.

   `Base:` For $n = 1$, $\sum_{k=0}^{n} \binom{s+k-1}{k} = \binom{s-1}{0} + \binom{s}{1} = 1 + s = \binom{s+1}{1}$

   `Induction:` Assume the claim holds for $n = N$, i.e. $\sum_{k=0}^{N} \binom{s+k-1}{k} = \binom{s+N}{N}$. We will show it's true for $n = N + 1$. Consider:

$$\sum_{k=0}^{N+1} \binom{s+k-1}{k} = \left[\sum_{k=0}^{N} \binom{s+k-1}{k}\right] + \binom{s+(N+1)-1}{N+1} = \binom{s+N}{N} + \binom{s+N}{N+1}$$

   where we've applied the inductive hypothesis. Now apply Pascal's identity:

$$\binom{s+N}{N} + \binom{s+N}{N+1} = \binom{s+N+1}{N+1}$$

   which is what we wanted to show.

3. **Structural induction [10 points]**

   For any real numbers a and b, the set $S_{a,b}$ is defined recursively by

   1: $(4, 3) \in S_{a,b}$

   2: If $(x, y) \in S_{a,b}$, then $(-x, -y) \in S_{a,b}$

   3: If $(x, y) \in S_{a,b}$, then $(ax - by, ay + bx) \in S_{a,b}$

   Suppose that a and b are real numbers such that $a^2 + b^2 = 1$.

   (a) Use structural induction to prove that $x^2 + y^2 = 5$ for any element (x, y) in S.

   [**Solution**]

   Note the correction to the problem reforms the claim as:

   $x^2 + y^2 = 25$ *for any element* (x, y) *in* S

   We will show this by induction on $S_{a,b}$

   `Base:` $(4, 3)$ is in $S_{a,b}$ and $4^2 + 3^2 = 16 + 9 = 25$

   `Induction:` Assume that the claim holds for some element (x, y) of $S_{a,b}$, i.e. $x^2+y^2 = 25$. We need to show that the claim holds for $(-x, -y)$ and $(ax - by, ay + bx)$.

   The claim clearly holds for $(-x, -y)$:

$$(-x)^2 + (-y)^2 = x^2 + y^2 = 25$$

Now consider $(ax - by, ay + bx)$.

$$\begin{aligned}(ax-by)^2 + (ay+bx)^2 &= (a^2x^2 - 2axby + b^2y^2) + (a^2y^2 + 2aybx + b^2x^2) \\ &= a^2(x^2+y^2) + b^2(y^2+x^2) \\ &= 25a^2 + 25b^2 \\ &= 25(a^2+b^2) \\ &= 25\end{aligned}$$

(b) What familiar geometric figure is S a subset of?

[Solution]

A circle of radius 5, centered at the origin.

(c) For some choices of a and b, S is a finite set. What constraints must a and b satisfy for this to be true? Informally explain why your answer is right.

[Solution]

Rule 3 in the definition of $S_{a,b}$ performs a rotation of the point (x, y) about the origin. Thus we'll have a finite number of points whenever repeated rotations lead to a mulitple of 360°; i.e. we rotate back to the point we started with. a and b determine the angle of each rotation.

Formally, this will happen when $i\theta = j(2\pi)$, where θ is the angle of rotation, i is a positive integer, and j is a natural number. Equivalently, $\theta = 2\left(\frac{j}{i}\right)\pi = 2k\pi$. The values of a and b that yield finite $S_{a,b}$ will be $\cos(2k\pi)$ and $\sin(2k\pi)$, respectively, for any positive rational k.

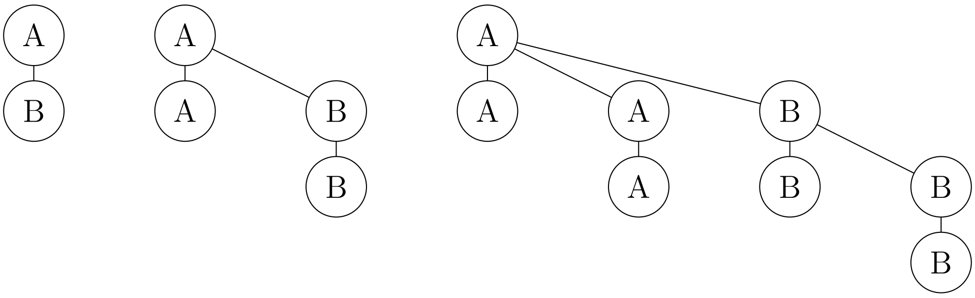
Note that rule 2 is equivalent to performing a 180° rotation, which will not affect our formalization above.

4. **Tree induction [14 points]**

A binomial tree of order k is defined recursively as follows:

- A single root node is a binomial tree of order 0.
- A binomial tree of order k consists of two binomial trees of order $k - 1$, with the root of the first connected as the rightmost child of the root of the second.

The following picture shows the binomial trees of order 1, 2, and 3. The labels on the nodes show how the larger tree is divided into two lower-order subtrees.



(a) Use induction on the order of the tree to prove that a binomial tree of order k has 2^k nodes.

[**Solution**]

By induction on the order, k.

`Base:` For $k = 0$, we have a binomial tree of order 0, which has $2^0 = 1$ nodes.

`Induction:` Assume that the claim holds for $k = K$: binomial trees of order K have 2^K nodes.

We will now prove the claim for $k = K + 1$. Consider a binomial tree of order $K + 1$: by definition, it is composed of two binomial trees of order K with no nodes otherwise added or removed. Thus this tree has exactly $2^K + 2^K = 2^{K+1}$ nodes, which is what we wanted to show.

(b) Use induction (again on the order of the tree) to prove that a binomial tree of order k has exactly $\binom{k}{i}$ nodes at level i. Hint: use induction on the order of the tree. For some randomly chosen level i, sum the numbers of nodes in the two trees.

Warning: in your inductive step you **must** divide the larger tree into smaller subtrees by taking it apart at the root. Do not try to graft things onto the bottom of a small tree to make a big one.

[**Solution**]

By induction on the order, k.

`Base:` For $k = 0$, we have a binomial tree of order 0. At level 0, this tree has $\binom{0}{0} = 1$ nodes.

`Induction:` Assume that the claim holds for $k = K$: binomial trees of order K have exactly $\binom{K}{i}$ nodes at level i.

We will show this is true for $k = K + 1$. Consider a binomial tree T of order $K + 1$. It contains a binomial tree of order K (call it S_1) with another binomial tree of order K attached as a child of the root (call it S_2). There are three parts of T to consider: the top level, the bottom level, and an arbitrary level in between. Say that h is the height of the tree.

At level 0, T has only a root, which is $1 = \binom{K+1}{0}$ node.

At level i, $0 < i < h$, T has $\binom{K}{i}$ nodes from S_1 and $\binom{K}{i-1}$ nodes from S_2. Adding these two together and applying Pascal's identity, we have:

$$\binom{K}{i} + \binom{K}{i-1} = \binom{K+1}{i}$$

At level h, S_1 doesn't have any more nodes and S_2 has exactly one node. Both $\binom{K}{K}$ and $\binom{K+1}{K+1}$ equal 1. To make this part fully precise, it looks like you would need to prove some additional property of these trees such as that the tree of order K has height K. This extra proof would not be hard, but it's more than we expecte for full credit on this problem.