CS 173: Discrete Structures, Fall 2009
Homework 9

This homework contains 4 problems worth a total of 50 points. It is due on Friday, November 13 at noon. Put your homework in the appropriate dropbox in the Siebel basement.

1. **Counting [18 points]**

   Solve the following counting problems, providing brief explanations and/or work to justify your answers. You do not need to multiply out factorials or trying to simplify formulas to reach your final answer. For example, $\frac{10!}{6!}$ would be fine as an answer.

   (a) Central Plains Pizza offers three choices for the diameter, two choices for crust thickness, and 12 choices of topping. Suppose that you limit yourself to 5 or fewer toppings. How many different pizzas can you construct?

   (b) How many distinct strings can you make out of the characters in the word "potato-soup"? (Include the hyphen as a character.)

   (c) You need to form a battle group of 11 made up of orcs, elves, and goblins. In how many ways can you choose the composition of your battle group?

   (d) You need to form a battle group consisting of an orc, an elf, and a goblin whose total strength is 12. The strength of each creature is an integer between 1 and 10 and the strength of the group is the sum of the individual strengths. In how many ways can you construct your battle group?

   (e) You are out in the middle of the cornfields, where the (dirt) roads all run north-south or east-west, evenly spaced at one road per mile. If you start at a road intersection and wish to get to the intersection 5 miles to the north and 7 miles east, how many different routes could you take?

   (f) Your latest cheapo cell phone keyboard only includes the uppercase alphabet (26 characters total). How many 12-character strings can you type that start with ST and contain no more than three T's?

2. **Combinatorial formula [8 points]**

   Use induction and Pascal's identity to prove the following formula holds for any positive integers s and n:

$$\sum_{k=0}^{n} \binom{s+k-1}{k} = \binom{s+n}{n}$$

3. **Structural induction [10 points]**

For any real numbers a and b, the set $S_{a,b}$ is defined recursively by

1: $(4, 3) \in S_{a,b}$
2: If $(x, y) \in S_{a,b}$, then $(-x, -y) \in S_{a,b}$
3: If $(x, y) \in S_{a,b}$, then $(ax - by, ay + bx) \in S_{a,b}$

Suppose that a and b are real numbers such that $a^2 + b^2 = 1$.

(a) Use structural induction to prove that $x^2 + y^2 = 5$ for any element (x, y) in S.

(b) What familiar geometric figure is S a subset of?

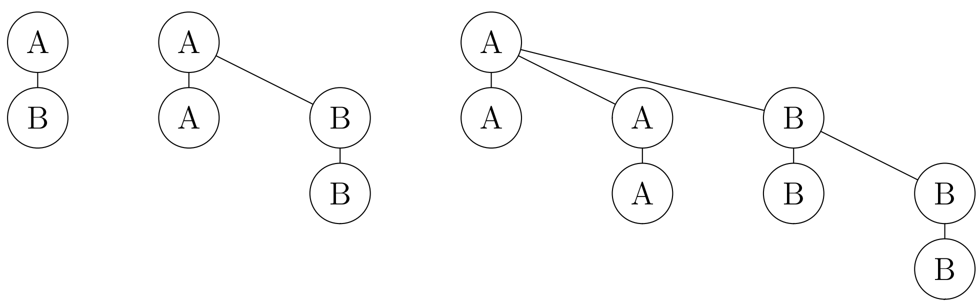
(c) For some choices of a and b, S is a finite set. What constraints must a and b satisfy for this to be true? Informally explain why your answer is right.

4. **Tree induction [14 points]**

A binomial tree of order k is defined recursively as follows:

- A single root node is a binomial tree of order 0.
- A binomial tree of order k consists of two binomial trees of order $k - 1$, with the root of the first connected as the rightmost child of the root of the second.

The following picture shows the binomial trees of order 1, 2, and 3. The labels on the nodes show how the larger tree is divided into two lower-order subtrees.



(a) Use induction on the order of the tree to prove that a binomial tree of order k has 2^k nodes.

(b) Use induction (again on the order of the tree) to prove that a binomial tree of order k has exactly $\binom{k}{i}$ nodes at level i. Hint: use induction on the order of the tree. For some randomly chosen level i, sum the numbers of nodes in the two trees.

Warning: in your inductive step you **must** divide the larger tree into smaller subtrees by taking it apart at the root. Do not try to graft things onto the bottom of a small tree to make a big one.