CS 173: Discrete Structures, Fall 2009
Homework 6 Solutions

This homework contains 5 problems worth a total of 50 regular points, plus a 5-point bonus problem.

1. **Recursive definition [12 points]**

   Give a simple closed-form definition for each of the following recursively-defined subsets of the real plane. Give both a precise definition using set-builder notation and also an informal geometrical description using a picture and/or words.

   (a) The set T defined by:

   i. $(2, 2) \in T$
   ii. If $(x, y) \in T$, then $(y, x) \in T$.
   iii. If $(x, y) \in T$, then $(-x, y) \in T$.

   [**Answer:**] $T = \{(2, 2), (-2, 2), (2, -2), (-2, -2)\}$. That is, T is the set of points that define the corners of a square centered at the origin with side-length 2.

   (b) The set S defined by:

   i. $(2, 2) \in S$
   ii. If $(x, y) \in S$, and n is any positive integer, then $(ny, nx) \in S$.

   [**Answer:**] $S = \{(2n, 2n) \mid n \in \mathbb{Z}^+\}$. That is, S is the set of positive even integer points along the line $y = x$.

   (c) The set M defined by:

   i. $(2, 2) \in M$
   ii. If $(x, y) \in M$, then $(y, x) \in M$.
   iii. If $(x, y) \in M$, then $(-x, y) \in M$.
   iv. If $(x, y) \in M$, and α is any real number ≥ 1, then $(\alpha y, \alpha x) \in M$.

   [**Answer:**] $M = \{(a, a) \mid a \in \mathbb{R} \text{ and } |a| \geq 2\} \cup \{(-a, a) \mid a \in \mathbb{R} \text{ and } |a| \geq 2\}$. That is, M is the set of points along the lines $y = x$ and $y = -x$ where $|x| \geq 2$.

2. **Induction [10 points]**

Use induction to prove that the following equation holds for all positive integers n:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

[**Answer:**] We prove this by induction on n.

BASE CASE: Note that when $n = 1$, both sides of the equation equal $1/2$.

INDUCTION HYPOTHESIS: Let $N \in \mathbb{Z}$ and suppose that the statement is true for $n = N - 1$. That is, suppose that

$$\sum_{k=1}^{N-1} \frac{1}{k(k+1)} = \frac{N-1}{N}$$

INDUCTIVE STEP: We want to show that this statement holds for $n = N$ when the inductive hypothesis is true. We extract the last value from the sum to obtain the following:

$$\sum_{k=1}^{N} \frac{1}{k(k+1)} = \frac{1}{N(N+1)} + \sum_{k=1}^{N-1} \frac{1}{k(k+1)}$$

Then, we apply the inductive hypothesis:

$$\sum_{k=1}^{N} \frac{1}{k(k+1)} = \frac{1}{N(N+1)} + \frac{N-1}{N}$$

Multiplying top and bottom of $\frac{N-1}{N}$ by $N+1$, we can combine the terms to get

$$\frac{1 + (N+1)(N-1)}{N(N+1)} = \frac{N^2}{N(N+1)} = \frac{N}{N+1}$$

as desired.

3. **Son of induction [10 points]**

Define a function $g : \mathbb{N} \to \mathbb{R}$ by

- $g(0) = 0$
- $g(n) = n + 3g(n-1)$ for all integers $n \geq 1$

(a) Calculate the next four values of g, i.e. $g(1)$, $g(2)$, $g(3)$, $g(4)$.

[**Answer:**]

- $g(1) = 1$
- $g(2) = 5$
- $g(3) = 18$
- $g(4) = 58$

(b) Use induction to prove that $g(n) = \frac{3^{n+1}-2n-3}{4}$ for every integer $n \geq 0$.

[**Answer:**] We prove this by induction on n.

Base Case: See part (a) and note that $g(0) = \frac{3-0-3}{4} = 0$ and $g(1) = \frac{9-2-3}{4} = 1$.

Inductive Hypothesis: Suppose that the statement is true for $n = N-1$, for some integer N. That is, suppose that $g(N-1) = \frac{3^N-2(N-1)-3}{4}$.

Inductive Step: We assume the inductive hypothesis, and want to show that the formula holds for $n = N$. By the definition of g, we have the following:

$$g(N) = N + 3g(N-1)$$

From the inductive hypothesis, we get

$$g(N) = N + 3\frac{3^N - 2(N-1) - 3}{4} = \frac{4N + 3^{N+1} - 6N + 6 - 9}{4} = \frac{3^{N+1} - 2N - 3}{4}$$

as desired.

4. **Induction on congruences** [6 points]

We've proved that if $a \equiv b \pmod{p}$ and $c \equiv d \pmod{p}$, then $a + c \equiv b + d \pmod{p}$ and $ac \equiv bd \pmod{p}$, for any integers a, b, c, and d and any positive integer p. Using one or both of these facts and induction, prove the following claim:

> For any integers a and b and any positive integers n and p, if $a \equiv b \pmod{p}$, then $a^n \equiv b^n \pmod{p}$.

[**Answer:**] Suppose that a, b, n, and p are integers, with n and p positive and suppose that $a \equiv b \pmod{p}$. We prove that $a^n \equiv b^n \pmod{p}$ by induction on n.

Base Case: When $n = 1$, the claim is obvious. When $n = 2$, we use the fact that $ac \equiv bd \pmod{p}$ when $a \equiv b \pmod{p}$ and $c \equiv d \pmod{p}$, and let $c = a, b = d$ to obtain that $a^2 \equiv b^2 \pmod{p}$.

INDUCTIVE HYPOTHESIS: Suppose that the claim is true when $n = N - 1$ for $N \in \mathbb{Z}$. That is, suppose that $a^{N-1} \equiv b^{N-1} \pmod{p}$.

INDUCTIVE STEP: We want to show that $a^N \equiv b^N \pmod{p}$ assuming the inductive hypothesis is true. If we let $c = a^{N-1}$ and $d = b^{N-1}$, we can apply the fact that $ac \equiv bd \pmod{p}$ and the inductive hypothesis to obtain the result.

5. **Function composition and strictly increasing [12 points]**

   In this problem, assume that the domain of each function is a subset of the real numbers, e.g. the reals, the rationals, the even integers. And assume the same for the co-domain. With this assumption, recall that a function f is strictly increasing if $x < y$ implies that $f(x) < f(y)$.

   (a) If $f : A \to B$ is a strictly increasing function, prove that the reverse implication holds. That is, that $f(x) < f(y)$ implies that $x < y$. Hint: proof by contrapositive is one good approach.

   [**Answer:**] We prove the contrapositive. That is, suppose that $y \leq x$; we want to show that $f(y) \leq f(x)$. If $x = y$, then clearly $f(x) = f(y)$, and since f is strictly increasing, if $y < x$ then $f(y) < f(x)$, by definition. Thus, if $y \leq x$, $f(y) \leq f(x)$.

   (b) Suppose that $f : A \to B$ and $g : B \to C$. Prove that if g and $g \circ f$ are both strictly increasing, f is also strictly increasing. (Hint: use the result from part a.)

   [**Answer:**] Let $h = g \circ f$. Note that h is a function from A to C. Let $a, a' \in A$, and let $a < a'$. We are given that h is strictly increasing, which means that $h(a) < h(a')$.

   Also, since g is strictly increasing, by part (a) we know that $g(x) < g(y)$ implies that $x < y$. Let $x = f(a)$ and $y = f(a')$. Then, this result says that if $h(a) = g(f(a)) < g(f(a')) = h(a')$, then $f(a) < f(a')$.

   Thus, we have shown that $a < a'$ implies $f(a) < f(a')$, and thus f is strictly increasing.

   (c) If f and $g \circ f$ are both strictly increasing, it is not necessarily the case that g is also strictly increasing. Give a counter-example that show why this implication fails.

   [**Answer:**] Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined as $f(x) = x$, and $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = x^2$. Then $g \circ f : \mathbb{R}^+ \to \mathbb{R}$ is strictly increasing (as is f), but g is not strictly increasing.

6. **BONUS: Induction and geometry [5 point]**

   Claim: Suppose that we draw any number of straight lines in the plane, with the restriction that no two are parallel and no intersection point belongs to more than two lines. The lines divide up the plane into a set of regions. We're going to color each region red or green. It is possible to choose the color for each region so that adjacent regions never have the same color.

   In this claim, "adjacent" means that the two regions share an edge. Two regions touching at a single point are not considered adjacent. Also remember that a "line" (as opposed to a "line segment") goes off to infinity at both ends. So each line cuts the plane into two pieces.

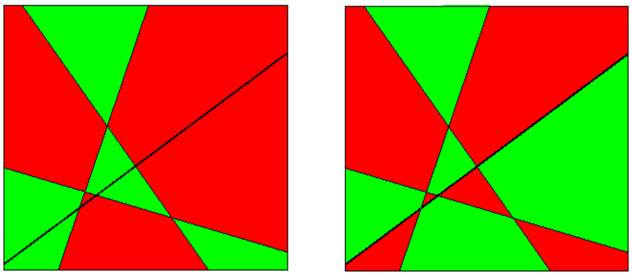

Figure 1: We consider the above configuration of lines. We remove the bold line, and create a proper coloring of the configuration with the line removed (left-hand side). Then, we reintroduce the bold line and swap all colors on the right-hand side to construct a new proper coloring.

[**Answer:**] Let n be the number of lines drawn in the plane; we prove this by induction on n. Also define a "proper" coloring to be one in which no region is adjacent to a different region with the same color.

Base Case: Note that if $n = 1$, there is exactly one line dividing the plane; we can color one side of it red and the other side green.

Inductive Hypothesis: Suppose that for $n = N - 1$, the claim holds, for some integer N. That is, suppose that if we draw $N - 1$ non-parallel lines in the plane such that no more than 2 lines intersect at a given point, then we can find a proper coloring for this configuration.

Inductive Step: We want to show that if the inductive hypothesis is true for $N-1$ lines that we can find a proper coloring using N lines. Suppose we are given some configuration of N non-parallel lines where no three lines intersect at a point. Pick an arbitrary line and remove it from the configuration. By the inductive hypothesis, we can find a proper coloring for this reduced configuration.

Now, add the line back into the configuration. Notice that the only regions that are affected by this addition are the ones which are split by the addition of this line. To fix this, select one side of the line, and swap the color of every region on this side (see figure 1). Note the following:

- Since we had a proper coloring on this side of the line before, if we swap the colors of every region, this remains proper.
- For each region split by the re-addition of the N^{th} line, the color on one side of the line is now different than the color on the other side of the line.
- Every color on the side of the line that we left alone is still not adjacent to another region of the same color, since we had a proper coloring before and we changed nothing on this side of the line.

Thus, we have now constructed a proper coloring on a configuration with N lines, and the theorem is proved.