CS 173: Discrete Structures, Fall 2009
Homework 5

This homework contains 5 problems worth a total of 50 regular points. It is due on Friday, October 9 at noon. Put your homework in the appropriate dropbox in the Siebel basement.

1. **Functions [8 points]**

   For each of the following functions, give the following information: what is its co-domain? what is its image? is the function onto? is the function one-to-one?

   (a) $f : \mathbb{C} \to \mathbb{R}$ such that $f(x + yi) = x + y$

   (b) $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^3 + 7$

   (c) $h : \mathbb{Z}^2 \to \mathbb{Z}$ such that $h(x, y) = 2^x 3^y$

   (d) $k : [0, 1] \to \mathbb{R}^2$ such that $k(x) = x(3, 4) + (1 - x)(1, 2)$. (The product of a number a and a 2D point (x, y) is (ax, ay) i.e. multiply both coordinates by a.)

   If you can't immediately see the answer for (d), pick some convenient input values and compute their output values. Try plotting the output values in 2D.

2. **Nested quantifiers [8 points]**

   State whether the following propositions are true or false, and briefly (but clearly!) explain why.

   (a) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}, y^3 \leq x$

   (b) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N}, xy = x$

   (c) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{Q}, |x - y| \leq 0.01$

   (d) $\exists y \in \mathbb{Q},\ \forall x \in \mathbb{R}, |x - y| \leq 0.01$

3. **Proving a set relation [10 points]**

   (a) Prove the following set inclusion by choosing an element from the smaller set and showing that it's in the larger set.

   For any sets A, B, and C, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

   (b) Show that the set inclusion doesn't hold in the other direction, by giving specific sets A, B, and C for which it fails.

   Hint: drawing yourself a Venn diagram may help you understand what's happening.

4. **Proofs with concrete functions [10 points]**

   Prove the following claims. Do this directly from the definitions of one-to-one and onto, plus high-school algebra. **Do not use calculus or theorems about increasing functions.**

   (a) Consider $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 2^{x+1}$. Show that g is one-to-one.

   (b) Consider $f : \mathbb{Z}^2 \to \mathbb{Z}$ such that $f(x, y) = xy + 27$. Show that f is onto.

5. **A curious bijection [14 points]**

   Mathematicians define two sets to be the same size if you can produce a bijection from one to the other. You might imagine that the set of pairs of natural numbers is larger than the natural numbers. However, it turns out that they are the same size.

   Let's define a function f from $\mathbb{N}^2$ to $\mathbb{N}$ as follows:

$$f(x, y) = \begin{cases} x + y^2 & \text{if } y \geq x \\ (x+1)^2 - (y+1) & \text{otherwise} \end{cases}$$

   Pretty mysterious, eh? I claim that f is actually a bijection and, thus, it has an inverse which is a function mapping $\mathbb{N}$ onto all of $\mathbb{N}^2$.

   (a) (3 points) Draw a picture of what the function does for pairs (x, y) such that $x + y \leq 4$. That is, draw a 2D table. At the 2D position corresponding to (x, y), write the value of the function f for input (x, y). That is, at location $(1, 2)$ in your picture, write the value 5.

   (b) What is the range of possible output values for an input pair (x, k) where $x \leq k$? That is, write formulas (depending only on k) for the largest and smallest output values.

   (c) What is the range of possible output values for an input pair (k, y) where $y < k$?

   (d) What is the preimage of the output value 17?

   (e) If the larger of the two input coordinates is k, how many different input values are possible? How does this compare to the number of different output values?

   (f) (3 points) Suppose that we have pairs (x, y) and (p, q) such that $\max(x, y) \neq \max(p, q)$. Explain why $f(x, y)$ cannot be equal to $f(p, q)$.

   This problem should give you a strong intuition for why f is both one-to-one and onto. To firm this up into a formal proof, we'd need to explain why f is one-to-one within each of the ranges of input values carved out in parts (b) and (c). I won't make you do this because it's a bit messy.