CS 173: Discrete Structures, Fall 2009
Homework 5, Solutions

This homework contains 5 problems worth a total of 50 regular points. It is due on Friday, October 9 at noon. Put your homework in the appropriate dropbox in the Siebel basement.

1. **Functions [8 points]**

   For each of the following functions, give the following information: what is its co-domain? what is its image? is the function onto? is the function one-to-one?

   (a) $f : \mathbb{C} \to \mathbb{R}$ such that $f(x + yi) = x + y$

   (b) $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^3 + 7$

   (c) $h : \mathbb{Z}^2 \to \mathbb{Z}$ such that $h(x, y) = 2^x 3^y$

   (d) $k : [0, 1] \to \mathbb{R}^2$ such that $k(x) = x(3, 4) + (1 - x)(1, 2)$. (The product of a number a and a 2D point (x, y) is (ax, ay) i.e. multiply both coordinates by a.)

   Solution:

   (a) The codomain is $\mathbb{R}$, the image is $\mathbb{R}$, the function is onto but not one-to-one.

   (b) The codomain is $\mathbb{R}$, the image is $\mathbb{R}$, the function is one-to-one and onto.

   (c) This is not a function. Observe that $h(-1, -1) = 1/6$, which is not an integer. The corrected version of the problem is:
   $h : \mathbb{N}^2 \to \mathbb{N}$ such that $h(x, y) = 2^x 3^y$
   Now, the codomain is $\mathbb{N}$, the image is $\{2^x 3^y | x, y \in \mathbb{Z}\}$ (which is the set of positive integers that do not have any prime factors other than 2 and 3), the function is one-to-one but not onto.

   (d) The codomain is $\mathbb{R}^2$, the image is the line segment in $\mathbb{R}^2$ between the points $(3, 4)$ and $(1, 2)$, the function is one-to-one but not onto.

2. **Nested quantifiers [8 points]**

   State whether the following propositions are true or false, and briefly (but clearly!) explain why.

   (a) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}, y^3 \leq x$

   (b) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N}, xy = x$

   (c) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{Q}, |x - y| \leq 0.01$

   (d) $\exists y \in \mathbb{Q},\ \forall x \in \mathbb{R}, |x - y| \leq 0.01$

   Solution:

   (a) True. Let $y = 0$ for non-negative values of x and $y = x$ for negative values of x.

(b) True. $x = 0$.

(c) True. Consider the decimal representation of x. It may be infinite for non-rational values of x. Let y be equal to x up to the 100^{th} digit after the decimal point and zero after that.

(d) False. We can get the counter example for any value of y by setting $x = y + 1$.

3. **Proving a set relation [10 points]**

(a) Prove the following set inclusion by choosing an element from the smaller set and showing that it's in the larger set.

For any sets A, B, and C, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

(b) Show that the set inclusion doesn't hold in the other direction, by giving specific sets A, B, and C for which it fails.

Solution:

(a) If $x \in (A \cup B) \cap C$ then $x \in C$ and $x \in A \cup B$. So $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \cap C)$. If $x \in B$ then $x \in B \cap C$ and so $x \in A \cup (B \cap C)$.

(b) Let $A = \{a\}$ and $B = C = \emptyset$. Then $(A \cup B) \cap C = \emptyset$ and $A \cup (B \cap C) = \{a\}$.

4. **Proofs with concrete functions [10 points]**

   Prove the following claims. Do this directly from the definitions of one-to-one and onto, plus high-school algebra. **Do not use calculus or theorems about increasing functions.**

   (a) Consider $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 2^{x+1}$. Show that g is one-to-one.

   (b) Consider $f : \mathbb{Z}^2 \to \mathbb{Z}$ such that $f(x, y) = xy + 27$. Show that f is onto.

   Solution:

   (a) Assume $g(x) = g(y)$. So,

   $$2^{x+1} = 2^{y+1} \Rightarrow x + 1 = y + 1 \Rightarrow x = y$$

   (b) Let $z \in \mathbb{Z}$. We like to show there exists an element $(x, y) \in \mathbb{Z}^2$ whose image is z. That is $xy + 27 = z$. By setting $x = 1$ we get the following solution $y = z - 27$. Therefore $(1, z - 27)$ is the preimage of z.

5. **A curious bijection [14 points]**

   Mathematicians define two sets to be the same size if you can produce a bijection from one to the other. You might imagine that the set of pairs of natural numbers is larger than the natural numbers. However, it turns out that they are the same size.

   Let's define a function f from $\mathbb{N}^2$ to $\mathbb{N}$ as follows:

   $$f(x, y) = \begin{cases} x + y^2 & \text{if } y \geq x \\ (x+1)^2 - (y+1) & \text{otherwise} \end{cases}$$

   Pretty mysterious, eh? I claim that f is actually a bijection and, thus, it has an inverse which is a function mapping $\mathbb{N}$ onto all of $\mathbb{N}^2$.

   (a) (3 points) Draw a picture of what the function does for pairs (x, y) such that $x + y \leq 4$. That is, draw a 2D table. At the 2D position corresponding to (x, y), write the value of the function f for input (x, y). That is, at location $(1, 2)$ in your picture, write the value 5.

   (b) What is the range of possible output values for an input pair (x, k) where $x \leq k$? That is, write formulas (depending only on k) for the largest and smallest output values.

   (c) What is the range of possible output values for an input pair (k, y) where $y < k$?

   (d) What is the preimage of the output value 17?

   (e) If the larger of the two input coordinates is k, how many different input values are possible? How does this compare to the number of different output values?

   (f) (3 points) Suppose that we have pairs (x, y) and (p, q) such that $\max(x, y) \neq \max(p, q)$. Explain why $f(x, y)$ cannot be equal to $f(p, q)$.

Solution:

(a) See Figure 1. The left and right side figures correspond to the cases $x + y \leq 4$ and $\max(x, y) \leq 4$, respectively.

(b) The fact that $0 \leq x \leq k$ implies $f(x, k) = x + k^2$. So $k^2 = f(0, k) \leq f(x, k) \leq f(k, k) = k + k^2$.

(c) The fact that $0 \leq y < k$ implies $f(k, y) = (k+1)^2 - (y + 1)$. So $(k + 1)^2 - k = f(k, k-1) \leq f(k, y) \leq f(k, 0) = (k + 1)^2 - 1$.

(d) It is $(1, 4)$.

(e) The number of possible input values is $2k + 1$, because the smaller value has k choices $0, 1, \ldots, k - 1$ and the order of the values is important. Finally there is one more option (k, k) to count. Parts (b) and (c) imply that the number of possible outputs is exactly $2k + 1$. The output can be any number in the range $[k^2, (k + 1)^2)$ which is contains $(k + 1)^2 - k^2 = 2k + 1$ numbers.

(f) Parts (b) and (c) imply that these two numbers are coming from different non-overlapping ranges, namely $[(max(x, y))^2, (max(x, y){+}1)^2)$ and $[(max(p, q))^2, (max(p, q){+}1)^2)$. To observe it assume (without loss of generality) that $max(x, y) < max(p, q)$. Since both of these maximum values are natural numbers we have $max(x, y){+}1 \leq max(p, q)$, which means $(max(x, y) + 1)^2 \leq (max(p, q))^2$.

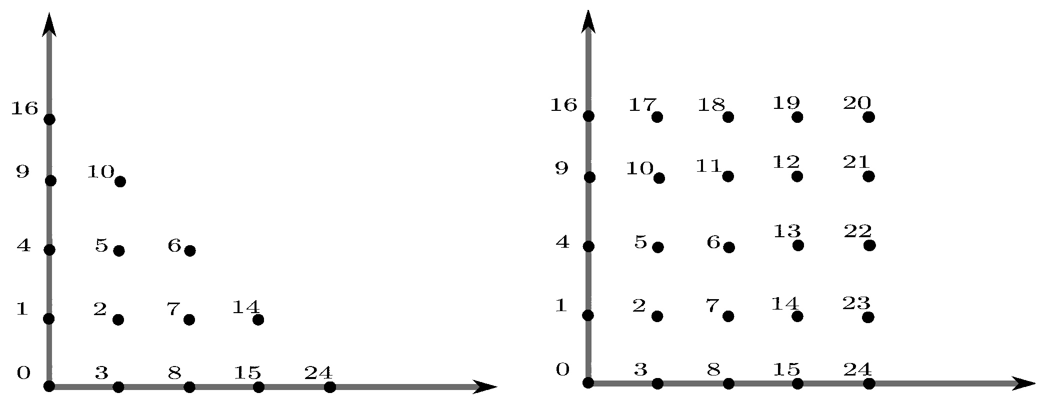

Figure 1: Problem 5(a)