CS 173: Discrete Structures, Fall 2009
Homework 4

This homework contains 6 problems worth a total of 50 regular points plus 5 bonus points. It is due on Friday, September 25th at noon. Turn in your homework in our new homework dropboxes, in the Siebel basement corridor (just east of the big window lounge area). The issue last week with the doors of the boxes has supposedly been fixed.

Starting with this homework, we will take one point off your homework score if your name and the day and time of your section are not clearly visible on the top sheet of your homework. If you want to include the problem statements with your homework, please staple them to the back (not the front) of your solutions.

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1. **[16 points] Sets**

   Define the following sets

$$\begin{aligned}
A &= \{\{\text{Elm}\}, \{\text{Pine}\}\} \\
B &= \{\text{Elm}, \text{Oak}, \text{Maple}\} \\
C &= \{\text{Elm}, \text{Vine}, \text{Birch}, \text{Maple}\} \\
D &= \{\text{Tree}, \text{Disease}, \text{Street}\}
\end{aligned}$$

   For each of the following expressions, list the elements of the set or calculate the value (as appropriate).

   (a) $\{(x, y) \in \mathbb{Z}^2 \mid x \geq 0 \text{ and } y \geq 0 \text{ and } x + y = 3\}$

   (b) $\{x \in \mathbb{Z} \mid -20 \leq x \leq 20 \text{ and } x \equiv 2 \pmod 7\}$

   (c) $\{X \in \mathbb{P}(C) \ : \ |X| \text{ is even}\}$

   (d) $A \cap B$

   (e) $A \cap \mathbb{P}(B \cap C)$

   (f) $(B \cap C) \times D$

   (g) $|\mathbb{P}(C \times D)|$

   (h) $|\mathbb{P}(B \cap D)|$

2. **[8 points] Euclidean algorithm**

   (a) Trace the execution of the Euclidean algorithm (lecture 10 or p 229 in Rosen) on the inputs 391 and 2380. That is, give a table showing the values of the main variables (x, y, r) for each pass through the loop. Explicitly indicate what the output value is.

   (b) The pseudo code for the Euclidean algorithm presented in lecture handles only positive number inputs, despite the fact that the definition of gcd also works if the inputs are negative or (in some cases) zero. Modify the recursive version of the pseudocode so that it can take any two integers as input. Your pseudocode should output an answer that follows our mathematical definition of gcd. The pseudocode can signal errors (e.g. illegal inputs) using the error command, e.g.

```
if (excessive_pressure)
   error ``Please don't squeeze the tomatoes.''
```

3. [**8 points**] Pseudocode

   I found the following uncommented pseudocode in Professor Snape's lab notebook.

```
procedure foo(n,m:  natural numbers)
   if (m = 0) return 1
   else return n*foo(n,m-1)

procedure bar(n: natural number)
   p := 0
   for i := 0 to n
       p := p + foo(n,3)
   return p
```

   (a) Describe the output of foo, as a (simple) function of the inputs n and m.

   (b) Describe the output of bar, as a function of the input n. Give your answer as a summation and also in closed form (equation that doesn't involve a summation).

4. [**10 points**] Direct proof using congruence mod k

   In the book, you will find several equivalent ways to define congruence mod k. For this problem, use the following definition: for any integers x and y and any positive integer m, $x \equiv y \pmod{m}$ if there is an integer k such that $x = y + km$.

   Using this definition prove that, for all integers x, y, p, q and m, with $m > 0$, if $x \equiv p \pmod{m}$ and $y \equiv q \pmod{m}$, then $x^2 + xy \equiv p^2 + pq \pmod{m}$.

5. **[8 points] Proof using inequalities**

   Prove the following claims using direct proof.

   (a) For any integers m and k, if $0 < \frac{1}{k} < m$ then $\frac{m}{m^2+1} < k$.

   (b) For any integers m and k, if $k \leq 7$ and $0 < m - 3 \leq \frac{k}{7}$, then $m^2 - 9 \leq k$.

6. **[5 points] Bonus proof**

   (This is a bonus problem. That is, if you have time to do it, it will add a few bonus points to your homework average. But it's ok to skip it if you have run out of time or interest.)

   Show that if x, y and m are integers with $m \geq 2$, then if $x \equiv y \pmod{m}$ then $gcd(x, m) = gcd(y, m)$.