CS 173: Discrete Structures, Fall 2009
Homework 3 Solutions

This homework contains 4 problems worth a total of 40 points. It is due on Friday, September 18th at noon. Turn in your homework in our new homework dropboxes, in the Siebel basement corridor (just east of the big window lounge area). The boxes will be labelled: put yours in the correct box for your discussion section.

Notice that you must used the stated proof method in problems 3 and 4, even though that may not be the only way to prove the claims. This is because the main point of these problems is to learn how to use these proof techniques.

---

1. [**10 points**] **Thinking about GCD**

   Consider the following two claims. For each, determine whether it's true or false. Justify your answer with a counter-example (if the claim is false) or an explanation in terms of prime factors (if the claim is true).

   (a) For any positive integers p, q, and r, if $\gcd(p, q) = 1$ and $\gcd(q, r) = 1$, then $\gcd(p, r) = 1$.

   [**Answer**]
   False. Consider $p = 2$, $q = 3$, and $r = 4$. $\gcd(p, q) = 1$ and $\gcd(q, r) = 1$, but $\gcd(p, r) = 2$.

   (b) For any positive integers p, q, and r, if $\gcd(p, q) = 1$ and $\gcd(p, qr) = 1$, then $\gcd(p, r) = 1$.

   [**Answer**]
   True. If $\gcd(p, q) = \gcd(p, qr) = 1$, then p and qr do not have prime factors in common. Since qr is q and r multiplied together, the prime factors of qr are just those of q plus those of r. Thus p shares no common factors with q or r, so the $\gcd(p, r) = 1$.

   Hint: try putting in some concrete values for p, q, and r. This may help you figure out whether each claim is true or false, and get a start on understanding why.

2. [**10 points**] **Proof with divides**

   Prove the following claim directly from the definition of "divides" (i.e. don't use facts about divides proved in class or the book). A direct proof should work.

   Claim: For any integers p, q, and r, p non-zero, if $p \mid 3q$ and $3q \mid r$, then $p \mid 3q + r$.

[**Answer**]
By the definition of divides, $p|3q$ means $pm = 3q$ for some $m \in \mathbb{Z}$, and $3q|r$ means $3qn = r$ for some $n \in \mathbb{Z}$. (Note that m and n are *different* variables!)

Now consider $3q + r$ and substitute to create an expression that's a multiple of p:

$$3q + r = 3q + 3qn = 3q(1 + n) = pm(1 + n)$$

Since $m(1 + n)$ is also an integer, we have $p|(3q + r)$ by definition, which is what we wanted to show.

3. [**10 points**] **Proof by contrapositive**

   Consider the following claim:

   > Claim: For any integers m and n, if $7m + 5n = 147$, then m is odd or n is odd.

   (a) State the converse of the claim.

   [**Answer**]
   The converse is

   > For any integers m and n, if m is odd or n is odd, then $7m + 5n = 147$.

   (b) State the contrapositive of the claim.

   [**Answer**]
   The contrapositive is

   > For any integers m and n, if m is even and n is even, then $7m + 5n \neq 147$.

   Note that both substatements have been negated.

   (c) Use proof by contrapositive to prove the claim.

   [**Answer**]
   We will prove the contrapositive of the claim: For any integers m and n, if m is even and n is even, then $7m + 5n \neq 147$.

   By the definition of evenness, $m = 2i$ and $n = 2j$ for some $i, j \in \mathbb{Z}$. Consider that $7m + 5n = 7(2i) + 5(2j) = 2(7i + 5j)$ is even, by definition. But since 147 is odd and a number cannot be both odd and even, 147 cannot equal $7m + 5n$. This is what we wanted to show.

4. **[10 points] Proof by contradiction**

Consider the following claim:

Claim: $\sqrt{2} + \sqrt{6} < \sqrt{15}$

(a) State the negation of the claim.

[Answer]

The negation is

$$\sqrt{2} + \sqrt{6} \geq \sqrt{15}$$

(b) Use proof by contradiction to prove the claim.

[Answer]

We will show that the opposite is impossible. Assume that $\sqrt{2} + \sqrt{6} \geq \sqrt{15}$. $(\sqrt{2} + \sqrt{6})^2 \geq (\sqrt{15})^2$ by our assumption, and $(\sqrt{2} + \sqrt{6})^2 = 2 + 2\sqrt{12} + 6 = 8 + 2\sqrt{12} \geq \sqrt{15}^2 = 15$. By subtracting 8 from both sides, this implies that $2\sqrt{12} \geq 7$, which we can see is false by squaring both sides: $(2\sqrt{12})^2 = 4 * 12 = 48 \ngeq 7^2 = 49$. Thus our assumption was incorrect, and the claim is true by contradiction.