CS 173: Discrete Structures, Fall 2009
Homework 2 Solutions

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1. **[3 points] Translating notation into English**

   Suppose we define:

   - $C(x)$ is "x grew up in Chicago."
   - $D(x)$ is "x drives well."
   - $M(x)$ is "x is taking CS 173."
   - $K(x)$ is "x is taking CS 225."
   - S is the set of all students.

   Translate the following into English:

   $$\forall y \in S, (M(y) \wedge \neg K(y)) \rightarrow (D(y) \vee \neg C(y))$$

   Solution: If a student is taking CS 173 and not taking CS 225, then he drives well or did not grow up in Chicago.

2. **[9 points] Proofs with propositional equivalences**

   Prove that the following pairs of expressions are logically equivalent, using the known equivalences given in Table 6 of Rosen, plus the fact that $p \rightarrow q$ is logically equivalent to $\neg p \vee q$. (Table 6 is reproduced on the handout for lecture 4.)

   (a) $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$

   (b) $\neg p \rightarrow (q \rightarrow \neg r)$ and $r \rightarrow (q \rightarrow p)$

   (c) $(\neg p \vee q) \wedge (r \vee \neg q) \wedge (p \vee q)$ and $q \wedge r$

   Solution:

   (a)

   $(p \rightarrow q) \vee (p \rightarrow r) \equiv$ (Definition)
   $(\neg p \vee q) \vee (\neg p \vee r) \equiv$ (Commutative/Associative Law)
   $(\neg p \vee \neg p) \vee (q \vee r) \equiv$ (Idempotent Law)
   $\neg p \vee (q \vee r) \equiv$ (Definition)
   $p \rightarrow (q \vee r)$

   (b)

   $\neg p \rightarrow (q \rightarrow \neg r) \equiv$ (Definition)
   $\neg\neg p \vee (\neg q \vee \neg r) \equiv$ (Double Negation)
   $p \vee (\neg q \vee \neg r) \equiv$ (Associative/Commutative Laws)
   $\neg r \vee (\neg q \vee p) \equiv$ (Definition)
   $r \rightarrow (q \rightarrow p)$

(c)

$$
\begin{aligned}
&(\neg p \lor q) \land (r \lor \neg q) \land (p \lor q) \equiv && \text{(Commutative Law)}\\
&(q \lor \neg p) \land (q \lor p) \land (r \lor \neg q) \equiv && \text{(Distributive Law)}\\
&(q \lor (p \land \neg p)) \land (r \lor \neg q) \equiv && \text{(Negation Law)}\\
&(q \lor F)) \land (r \lor \neg q) \equiv && \text{(Identity Law)}\\
&q \land (r \lor \neg q) \equiv && \text{(Distributive Law)}\\
&(q \land r) \lor (q \land \neg q) \equiv && \text{(Negation Law)}\\
&(q \land r) \lor F) \equiv && \text{(Identity Law)}\\
&q \land r
\end{aligned}
$$

3. **[8 points] Negating things**

Give the negation of the following logical expressions, using logical equivalences to move the "not" operators onto the smallest elements possible. For example, to negate $\forall x, [P(x) \rightarrow Q(x)]$, we first negate the whole thing $\neg\forall x[P(x) \rightarrow Q(x)]$, then convert this to $\exists x[\neg(P(x) \rightarrow Q(x))]$, and finally to $\exists x[P(x) \land \neg Q(x)]$. (For simplicity, we've omitted the domains for the quantified variables.)

(a) $\forall x[\neg M(x) \rightarrow Q(x)]$

(b) $\exists y[P(y) \land (\neg M(y) \lor R(y))]$

(c) $\exists x[\neg(P(x) \land Q(x)) \lor (F(x) \rightarrow P(x)]$

(d) $\forall z[(P(z) \land \neg M(z)) \rightarrow \neg P(z)]$

Solution:

(a) (2 points)

$\neg(\forall x[\neg M(x) \rightarrow Q(x)]) \equiv$

$\exists x[\neg(\neg M(x) \rightarrow Q(x))] \equiv$

$\exists x[\neg M(x) \land \neg Q(x)]$

(b) (2 points)

$\neg(\exists y[P(y) \land (\neg M(y) \lor R(y))]) \equiv$

$\forall y[\neg(P(y) \land (\neg M(y) \lor R(y)))] \equiv$

$\forall y[\neg P(y) \lor \neg(\neg M(y) \lor R(y))] \equiv$

$\forall y[\neg P(y) \lor (M(y) \land \neg R(y))]$

(c) (2 points)

$\neg(\exists x[\neg(P(x) \land Q(x)) \lor (F(x) \rightarrow P(x))]) \equiv$

$\forall x[\neg(\neg(P(x) \land Q(x)) \lor (F(x) \rightarrow P(x)))] \equiv$

$\forall x[\neg\neg(P(x) \land Q(x)) \land \neg(F(x) \rightarrow P(x)))] \equiv$

$\forall x[(P(x) \land Q(x)) \land (F(x) \land \neg P(x))]$

(d) (2 points)

$\neg(\forall z[(P(z) \land \neg M(z)) \rightarrow \neg P(z)]) \equiv$

$\exists z[\neg((P(z) \land \neg M(z)) \rightarrow \neg P(z))] \equiv$

$\exists z[(P(z) \land \neg M(z)) \land P(z)]$

4. **[10 points] Direct proof and disproof**

   (a) Prove the following claim:

   Claim: For any integer k, if k is odd, then k^3 is odd.

   (b) Disprove the following claim, by giving a concrete counter-example. (You must use a specific counter-example. Don't try to write a general or abstract story about why this equation can't possibly hold.)

   Claim: For any integers p and q, $(p+q)^2 = p^2 + q^2$.

Solution:

(a) (7 points) Let k be an odd integer. Since k is odd it is equal to $2i+1$ for some integer i.
$k = 2i+1 \Rightarrow k^3 = (2i+1)^3 = (2i)^3+3(2i)^2+3(2i)+1 = 2(4i^3+6i^2+3i)+1 = 2i'+1$
Because i' is an integer we conclude that k^3 is an odd number.

(b) (3 points) If $p = q = 1$, then: $(p+q)^2 = (1+1)^2 = 4$, but, $p^2 + q^2 = 1^2 + 1^2 = 2$.

5. **[10 points] Another direct proof**

Suppose that (x, y) and (p, q) are two intervals of the real line. Let's define (x, y) to *contain* (p, q) if and only if $x \le p$ and $q \le y$. Using this definition, prove the following claim:

Claim: For any intervals of the real line (a, b), (c, d), and (e, f), if (a, b) contains (c, d) and (c, d) contains (e, f), then if (a, b) contains (e, f).

Solution:

Suppose that (a, b), (c, d), and (e, f) are real intervals, (a, b) contains (c, d) and (c, d) contains (e, f).

$((a, b)$ contains $(c, d)) \Rightarrow (a \le c) \wedge (d \le b))$.

$((c, d)$ contains $(e, f)) \Rightarrow (c \le e) \wedge (f \le d))$.

So, we have: $(a \le e) \wedge (f \le b)$, which means (a, b) contains (e, f)

6. **[10 points] Son of direct proof**

   (a) Prove the following claim:

   Claim 1: For any integer k, if $k > 4$ then $2k + 1 < k^2$.

   (b) Using your result from part (a), prove the following claim:

   Claim 2: For any integer k, if $k > 4$ and $k^2 < 2^k$, then $(k+1)^2 < 2^{k+1}$.

Solution:

(a) (4 points) Suppose that k is an integer, where $k > 4$. Since $k > 4$, $k^2 > 4k$. But $4k = 2k + 2k > 2k + 1$. So $k^2 > 2k + 1$.

(b) (6 points) Suppose that k is an integer, where $k > 4$ and $k^2 < 2^k$.
$(k+1)^2 = k^2 + 2k + 1 < k^2 + k^2 = 2 \cdot k^2 < 2 \cdot 2^k = 2^{k+1}$.