CS 173: Discrete Structures, Fall 2009
Homework 11

This homework contains 5 problems worth a total of 48 points. It is due on Wednesday, December 9th at noon. Put your homework in the appropriate dropbox in the Siebel basement.

1. **Equivalence relations [8 points]**

   Let's define the relation $\sim$ on $\mathbb{R}$ such that $x \sim y$ if and only if $x - y \in \mathbb{Z}$.

   - List three elements of $[1.3]$. Give a precise set-builder description of what's in $[1.3]$.
   - How many equivalence classes does $\sim$ have?
   - Prove that $\sim$ is an equivalence relation.

2. **Partial orders [8 points]**

   In your mind, picture $\mathbb{R}^2$ as the set of (open) intervals of the real line. Let's define a relation $\ll$ on $\mathbb{R}^2$ as follows. Intuitively, $(a, b) \ll (c, d)$ if the two intervals overlap with (a, b) starting no later than (c, d). Formally:

   $$(a, b) \ll (c, d) \text{ if and only if } a \leq c \text{ and } c < b \leq d$$

   - Prove that $\ll$ is antisymmetric.
   - Prove that $\ll$ is not a partial order, using a concrete counter-example to show that it lacks one of the properties required to be a partial order.

3. **Showing an operation is well-defined [10 points]**

   Let's define a relation $\sim$ on $\mathbb{R}^2 - \{(0, 0)\}$ by

   $$(x, y) \sim (p, q) \text{ if and only if } (x, y) = (\lambda p, \lambda q) \text{ for some positive real number } \lambda$$

   Then define an addition operation on the equivalence classes of this relation:

 $$[(x, y)] + [(p, q)] = [(xp - yq, xq + yp)]$$

   (a) Each equivalence class is a familiar geometrical object. Which one? Give a geometrical description of what this addition operation does.

   (b) Prove that this addition operation is well-defined. That is, show that, if you pick different representatives for the two input equivalence classes, the output value still lives in the same equivalence class.

4. **Graphs and recursion [12 points]**

   Let's define a new set of graphs X_n as follows:

   - X_1 consists of two vertices and no edges.
   - For every $k \geq 2$, X_k consists of a copy of X_{k-1} plus two additional vertices. There is an edge from each of the additional vertices to each vertex in the copy of X_{k-1}.

   For example, the following figure shows X_1, X_2, X_3, and X_4. A star marks the vertices in the copy of X_{k-1}.
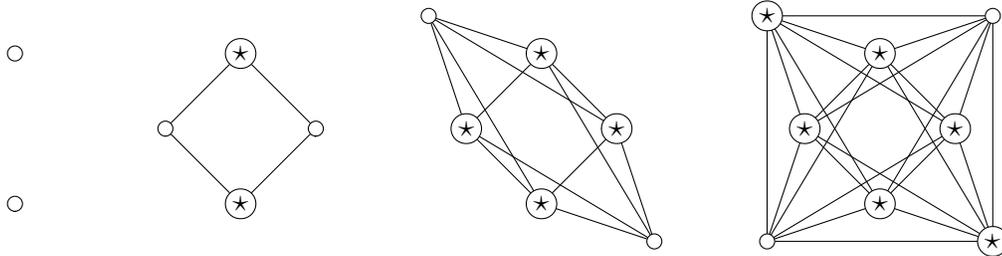
   Suppose that V_k and E_k are the number of vertices and edges in X_k.

   (a) Give a table showing the number of vertices and the number of edges in X_k, for k from 1 through 6.

   (b) Give a formula for V_k.

   (c) Write a recurrence (with base case) for E_k.

   (d) Solve the recurrence to get a closed-form expression for E_k.

   (e) The distance between two vertices a and b is the number of edges on the shortest path from a to b (which is zero if $a = b$). The diameter of a connected graph is the maximum distance between any two vertices. For $k \geq 2$, what is the diameter of X_k? Briefly explain why your answer is correct.

5. **Planar graphs [10 points]**

   See problem 1 for the definition of the graph X_k. Also notice that this depends on material which will be presented in lecture Friday.

   (a) Show that X_3 is planar by redrawing it so that pairs of edges never cross.

   (b) We will see in class (but perhaps not before you've started work on this homework) that a graph cannot be planar if it contains $K_{3,3}$ or K_5 as a subgraph. Show that X_4 is not planar by showing that it has a subgraph that is isomorphic to one of these two special graphs. The most effective way to do this is probably a combination of a labelled picture (start with what you did for part a) combined with some explanation.

   (c) Suppose that G is an undirected connected simple planar graph with 8 vertices, 2 of which have degree 4 and 6 of which have degree 3. How many edges does it have? How many regions does it have? (Hint: use Euler's formula.)