CS 173: Discrete Structures, Fall 2009
Homework 11 − Solutions

This homework contains 5 problems worth a total of 48 points.

1. **Equivalence relations [8 points]**

   Let's define the relation $\sim$ on $\mathbb{R}$ such that $x \sim y$ if and only if $x - y \in \mathbb{Z}$.

   - List three elements of $[1.3]$. Give a precise set-builder description of what's in $[1.3]$.
   - How many equivalence classes does $\sim$ have?
   - Prove that $\sim$ is an equivalence relation.

   Solution:

   - The real numbers 2.3, 3.3 and 1234.3 are elements of $[1.3]$.

     $$[1.3] = \{x | x \in \mathbb{R} \wedge (x - 1.3) \in \mathbb{Z}\}$$

   - It has uncountable number of classes. There is a class corresponding to each number in $[0, 1)$.
   - To prove that $\sim$ is an equivalence relation we need to show three facts:
     - It is Reflexive:
       $$\forall x \in \mathbb{R} : x - x \in \mathbb{Z} \Rightarrow x \sim x$$
     - It is Symmetric:
       $$\forall x, y \in \mathbb{R} : x \sim y \Rightarrow x - y \in \mathbb{Z} \Rightarrow y - x \in \mathbb{Z} \Rightarrow y \sim x$$
     - It is Transitive:
       $$\forall x, y, z \in \mathbb{R} : (x \sim y) \wedge (y \sim z) \Rightarrow (x - y \in \mathbb{Z}) \wedge (y - z \in \mathbb{Z})$$

       By adding two integers we get an integer. So:

       $$((x - y) + (y - z)) \in \mathbb{Z} \Rightarrow (x - z) \in \mathbb{Z} \Rightarrow x \sim z$$

2. **Partial orders [8 points]**

   In your mind, picture $\mathbb{R}^2$ as the set of (open) intervals of the real line. Let's define a relation $\ll$ on $\mathbb{R}^2$ as follows. Intuitively, $(a, b) \ll (c, d)$ if the two intervals overlap with (a, b) starting no later than (c, d). Formally:

   $(a, b) \ll (c, d)$ if and only if $a \leq c$ and $c < b \leq d$

   - Prove that $\ll$ is antisymmetric.

- Prove that $\ll$ is not a partial order, using a concrete counter-example to show that it lacks one of the properties required to be a partial order.

Solution:

- Let (a, b) and (c, d) be two elements of $\mathbb{R}^2$ such that $(a, b) \ll (c, d)$ and $(c, d) \ll (a, b)$. So, we have:
$$(a, b) \ll (c, d) \Rightarrow a \le c < b \le d$$
and
$$(c, d) \ll (a, b) \Rightarrow c \le a < d \le b$$
The inequalities $a \le c$ and $c \le a$ imply $c = a$. Similarly, $b \le d$ and $d \le b$ imply $b = d$. So, we have $(a, b) = (c, d)$.

- It is not Transitive. Here is a counter example: $(1, 5) \ll (2, 10)$ and $(2, 10) \ll (9, 11)$, but not $(1, 5) \ll (9, 11)$.

3. **Showing an operation is well-defined [10 points]**

   Let's define a relation $\sim$ on $\mathbb{R}^2 - \{(0, 0)\}$ by

   $(x, y) \sim (p, q)$ if and only if $(x, y) = (\lambda p, \lambda q)$ for some positive real number λ

   Then define an addition operation on the equivalence classes of this relation:

   $$[(x, y)] + [(p, q)] = [(xp - yq, xq + yp)]$$

   (a) Each equivalence class is a familiar geometrical object. Which one? Give a geometrical description of what this addition operation does.

   (b) Prove that this addition operation is well-defined. That is, show that, if you pick different representatives for the two input equivalence classes, the output value still lives in the same equivalence class.

   Solution:

   (a) Each equivalence class is a ray originated at $(0, 0)$ but excluding it. The output of the operation is the ray whose angle is the sum of the angles of the two input rays.

   To see this, notice that each equivalence class contains a point that has norm 1 (i.e. sum of the squares of the two coordinates is 1). Then its coordinates are the cosine and sine of the angle. And the equation in our operation is exactly the formulas for the cosine and sine of the sum of two angles. When one or both points has a norm that's not 1, the point $(xp - yq, xq + yp)$ produced by the raw equation moves outwards or inwards, compared to the two input points. But this doesn't change which ray the output point lives on. Or, said another way, our equivalence relation allows you to rescale the output point so as to give it norm 1 again.

(b) Let $(a, b) \in [(x, y)]$ and $(c, d) \in [(p, q)]$. So, there are positive values λ and λ' such that $(a, b) = \lambda(x, y)$ and $(c, d) = \lambda'(p, q)$.

$$(ac - bd, ad + bc) = (\lambda x \cdot \lambda' p - \lambda y \cdot \lambda' q, \lambda x \cdot \lambda' q + \lambda y \cdot \lambda' p) = \lambda\lambda'(xp - yq, xq + yp)$$

So, we have:

$$(ac - bd, ad + bc) \sim (xp - yq, xq + yp) \Rightarrow [(ac - bd, ad + bc)] = [(xp - yq, xq + yp)]$$

4. **Graphs and recursion [12 points]**

Let's define a new set of graphs X_n as follows:

- X_1 consists of two vertices and no edges.
- For every $k \geq 2$, X_k consists of a copy of X_{k-1} plus two additional vertices. There is an edge from each of the additional vertices to each vertex in the copy of X_{k-1}.

For example, the following figure shows X_1, X_2, X_3, and X_4. A star marks the vertices in the copy of X_{k-1}.
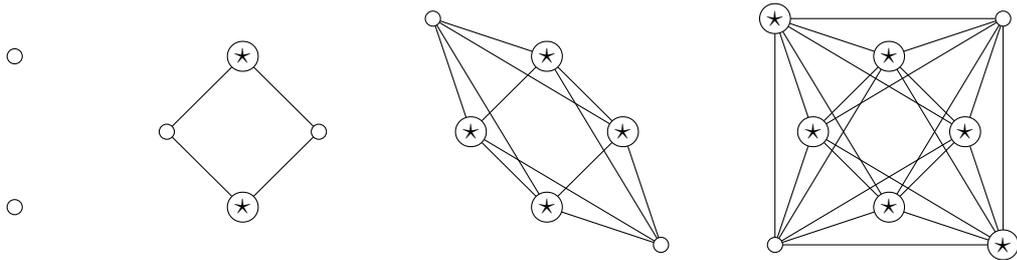
Suppose that V_k and E_k are the number of vertices and edges in X_k.

(a) Give a table showing the number of vertices and the number of edges in X_k, for k from 1 through 6.

(b) Give a formula for V_k.

(c) Write a recurrence (with base case) for E_k.

(d) Solve the recurrence to get a closed-form expression for E_k.

(e) The distance between two vertices a and b is the number of edges on the shortest path from a to b (which is zero if $a = b$). The diameter of a connected graph is the maximum distance between any two vertices. For $k \geq 2$, what is the diameter of X_k? Briefly explain why your answer is correct.

Solution:

(a)

k	1	2	3	4	5	6
$|V_k|$	2	4	6	8	10	12
$|E_k|$	0	4	12	24	40	60

(b) We know $V_k = V_{k-1} + 2$ and $V_1 = 2$. So $V_k = 2k$.

3

(c) $E_1 = 0$ and $E_k = E_{k-1} + 2V_{k-1} = E_{k-1} + 4(k-1)$.

(d) $E_k = E_{k-1} + 4(k-1) = E_{k-2} + 4(k-2) + 4(k-1) = 4(1+2+\cdots+k-1) = 2k(k-1)$

(e) The diameter of X_k is 2 when $k \geq 2$. The vertex set V_k is composed of V_{k-1} and two other vertices u and v. Any two vertices in X_{k-1} are both adjacent to u. The vertices u and v are both adjacent to any vertex of X_{k-1}. Finally, u and v are adjacent to all vertices in X_{k-1}. So, any two vertices can be connected by a path of length at most 2. On the other hand, there are non-adjacent vertices, namely those in V_1, which means the diameter is greater than 1.

5. **Planar graphs [10 points]**

See problem 1 for the definition of the graph X_k. Also notice that this depends on material which will be presented in lecture Friday.

(a) Show that X_3 is planar by redrawing it so that pairs of edges never cross.

(b) We will see in class (but perhaps not before you've started work on this homework) that a graph cannot be planar if it contains $K_{3,3}$ or K_5 as a subgraph. Show that X_4 is not planar by showing that it has a subgraph that is isomorphic to one of these two special graphs. The most effective way to do this is probably a combination of a labelled picture (start with what you did for part a) combined with some explanation.

(c) Suppose that G is an undirected connected simple planar graph with 8 vertices, 2 of which have degree 4 and 6 of which have degree 3. How many edges does it have? How many regions does it have? (Hint: use Euler's formula.)

Solution:

(a) See Figure 1. Vertices added at step i are labeled i and i'. Dashed and solid edges are added in steps 2 and 3, respectively.

(b) See Figure 2. Blue and red vertices show two partitions of the $K_{3,3}$. Dark edges show the edges of the $K_{3,3}$ subgraph.

(c) Let m be the number of edges. We know:

$$2m = \sum_{v \in V(G)} deg(v) = 2 \cdot 4 + 6 \cdot 3 = 26 \Rightarrow m = 13$$

Euler formula says: $n - m + f = 2$, where n and f are the number of vertices and regions, respectively. Particularly for G, we have: $8 - 13 + f = 2 \Rightarrow f = 7$.
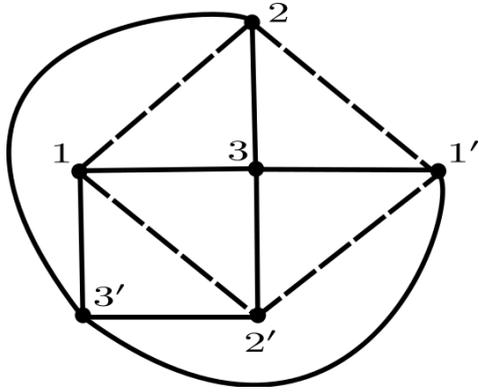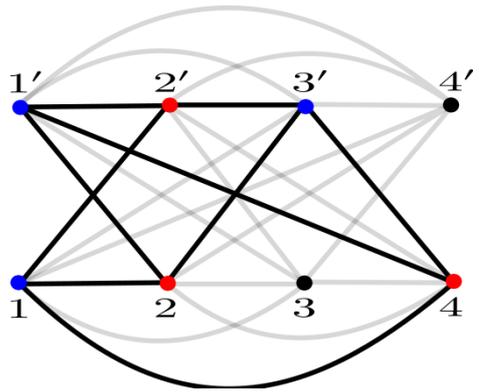
Figure 1: Planar X_3.

Figure 2: Non-planar X_4.