CS 173: Discrete Structures, Fall 2009
Homework 10 Solutions

This homework contains 4 problems worth a total of 34 points.

1. **Pigeonhole principle [6 points]**

   A software engineering at Snazzy Mobiles (a startup company) slept for 61 hours over 10 nights. Show that on some pair of consecutive nights, he slept at least 13 hours.

   [**Answer:**] Split the 10 nights into 5 pairs of consecutive days. Then we must divide 61 hours among these five pairs, but $61/5 > 12$, which means that (by the Pigeonhole Principle) at least one of these pairs has thirteen or more hours.

2. **Pigeonhole II [10 points]**

   Over a 30-day period, Frank learns one or more fiddle tunes each day. (Always an integer number of tunes on any given day.) Suppose that he learns no more than 45 tunes over the 30 days. Prove that there is a period of 1 or more consecutive days in which he learns exactly 14 tunes.

   Hint: Suppose a_i is the number of tunes learned in all the days up through and including i. Consider the integers $a_1, a_2, \ldots, a_{30}$, together with the integers $a_1+14, a_2+14, \ldots, a_{30}+14$.

   [**Answer:**] Let $b_1 = a_1 + 14, b_2 = a_2 + 14, ..., b_{30} = a_{30} + 14$. Then note that $a_1, ..., a_{30}$ is in the range $1, ..., 45$, and $b_1, ..., b_{30}$ in the range $15, ..., 59$. Thus, we have 60 total integers that we are trying to fit into the range $1, ..., 59$, but by the pigeonhole principle, two of them have the same value.

   Notice that a_i increases as i increases, because Frank learns at least one tune each day. So it's impossible for two of the a_i's to be equal. And, therefore, it's impossible for two of the b_i's to be equal. So our duplicate pair must contain one of the a_i's and one of the b_i's. That is, $a_i = b_j = a_j + 14$, for some i, j. Equivalently, $a_i - a_j = 14$, but by the definition of the a_i's, this implies that between day i and day j, Frank learns 14 tunes.

3. **Bernouilli Trials [8 points]**

   Marjorie is still a beginner at flute, so if she tries to play a high A, there is only a 0.3 probability that it will sound. If she makes 5 attempts to play a high A, then...

   (a) What is the chance that it will sound exactly three times?

   [**Answer:**] Let E be the event that the flute sounds exactly three times. Then we pick three of the five places for the flute to sound, and multiply by the probability of sounding three times and not sounding twice:

$$Pr(E) = \binom{5}{3}(0.3)^3(0.7)^2 \approx 0.13$$

(b) What is the chance that it will sound at least three times?

[**Answer:**] We sum over the probabilities that it sounds three, four, and five times:

$$\sum_{i=3}^{5} \binom{5}{i} (0.3)^i (0.7)^{5-i} \approx 0.16$$

(c) (4 points) What is the chance that it will sound three times in a row? Include cases in which it sounds more than three times, as long as at least three successes were consecutive. Briefly explain how you calculated your answer and why it's right. Hint: don't blindly apply the Bernouilli Trials formula. Reason from first principles.

[**Answer:**] We can simply list out all possible combinations of getting three successes in a row (let S and F denote success and failure, respectively):

SSSSS	SSSSF
FSSSS	SFSSS
SSSFS	SSSFF
FSSSF	FFSSS

Then we just calculate all probabilities of the above combinations:

$$Pr(\text{three consecutive successes}) = 0.3^5 + 4(0.3)^4(0.7) + 3(0.3)^3(0.7)^2 \approx 0.065$$

[**Alternate answer:**] We condition on the location of the first successful sound out of the three in a row. Note that the first successful sound can only happen on the first, second, or third attempt. Let S_3 be the event that we get three successes in a row, and let F_1, F_2, F_3 be the events that the first success of the three happens on attempt 1, 2, or 3 respectively. Then we need to calculate the following:

$$Pr(S_3) = Pr(S_3|F_1)Pr(F_1) + Pr(S_3|F_2)Pr(F_2) + Pr(S_3|F_3)Pr(F_3)$$

Applying the definition of conditional probability, we have that the above equals (where $Pr(X, Y)$ is the probability of X and Y happening at the same time.

$$Pr(S_3) = Pr(S_3, F_1) + Pr(S_3, F_2) + Pr(S_3, F_3)$$

If the first success occurs on the first attempt, we have $Pr(S_3, F_1) = (0.3)^3$; note that what happens on attempts four and five don't matter. Similarly, if the first success occurs on the second attempt, this means that the first attempt failed, which occurs with probability $Pr(S_3, F_2) = (0.7)(0.3)^3$. What happens on the last attempt is irrelevant. For the last attempt, note that we can either succeed on the first attempt and fail on the second, or fail on both the first and second attempts. Thus, $Pr(S_3, F_3) = (0.7)^2(0.3)^3 + (0.3)^4(0.7)$. Therefore, we have

$$Pr(S_3) = (0.3)^3(1 + 0.7 + (0.7)^2 + (0.7)(0.3)) \approx 0.065$$

4. **Expected value [10 points]**

Instead of just giving out chocolate on Halloween in the obvious way, your neighbor Mr. Egbert makes you draw cards to see how much he will give you. Mr. Egbert has a deck of 13 cards, with face values 1,2,...,13 and you get to choose a hand containing four cards.

For each hand h, you get $C(h)$ small chocolates, where $C(h)$ is the maximum face value of any of the cards in the hand. For example, you get 9 chocolates if you draw a hand containing 2, 5, 6, and 9.

- How big is our sample space (aka how many possible hands)?

  [**Answer:**] The sample space has $\binom{13}{4}$

- How many possible hands have k as the maximum value of the cards in the hand? (I.e. give a formula that depends on k.)

  [**Answer:**] There is one choice for the maximum card in the hand (k). When choosing the other three cards in the hand, we have $k-1$ values from which to pick them and the order in which we pick them is irrelevant. So there are $\binom{k-1}{3}$ hands with largest element k.

- What is the chance of drawing a hand with maximum value k?

  [**Answer:**] Divide the number of hands with maximum card k by the size of the sample space:

$$\frac{\binom{k-1}{3}}{\binom{13}{4}}$$

- How many chocolates can you expect to get out of this guy? (I.e. what is the expected value of C?)

  [**Answer:**] Sum over all values of k multiplied by the probabilty calculated above (note that when $k = 1, 2, 3$ the summation term is 0, which makes sense):

$$\sum_{k=1}^{13} k \frac{\binom{k-1}{3}}{\binom{13}{4}} = 11.2$$