1) (2 points each) Simplify the following expressions as much as possible, without using a calculator (either hardware or software). Do not approximate. Express all rational numbers as fractions. For complex numbers use i to represent the value $\sqrt{-1}$. Show your work.

(a) $\frac{(2^3 \times 2^5)^{10}}{512} = \frac{(2^{3+5})^{10}}{2^9} = \frac{2^{80}}{2^9} = 2^{71}$

(b) $\lfloor -4.6 \rfloor + \lceil -4.6 \rceil = -5 + -4 = -9$

(c) $(\log_2 13)(\log_{13} 2048) = (\log_2 13)(\log_{13} 2^{11}) = 11(\log_2 13)(\log_{13} 2)$. Using the change-of-base formula, we have $(\log_x a)(\log_a b) = \log_x b$, so this is $11 \log_2 2 = 11$

(d) $\frac{\log_3(81^k)}{7k} = \frac{k \log_3(3^4)}{7k} = \frac{4}{7}$

(e) $(1+i)(2-i)(3-i) = (2-i+2i-i^2)(3-i) = (3+i)(3-i) = (9-i^2) = 9+1 = 10$

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2)

(a) (3 points) Suppose that $x^2 + 3x - 18 < 0$. Which interval(s) of the real line must x belong to? Explain clearly why your answer is correct.

(b) (1 point) What are the formulas for the sin and cosine of the angle $x + y$ in terms of the sin and cosine of x and the sin and cosine of y?

(c) (3 points) Using your answer to (b), show that the following equality holds, where $x \in \mathbb{R}$, $n \in \mathbb{N}$, and i is $\sqrt{-1}$:

$$(\cos x + i \sin x)(\cos(nx) + i \sin(nx)) = \cos((n+1)x) + i \sin((n+1)x)$$

(a) Using the quadratic formula, we find the roots of the equation to be $x = -6$ and $x = 3$. Thus, we can rewrite the inequality as $(x+6)(x-3) < 0$. We seek a range of x-values such that this product is negative. Clearly this can occur when $x + 6 < 0$ and $x - 3 > 0$, or when $x + 6 > 0$ and $x - 3 < 0$, but not when both terms are positive or negative. Furthermore, x cannot be both greater than 3 and less than -6, so the only region for which the expression is negative is $-6 < x < 3$.

(b)
- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$ (Careful! Notice the switched sign on the right-hand side!)

(c) First, apply the distributive property; next, use the equations from part (b) to collect terms:

$$\begin{aligned}
(\cos x + i \sin x)(\cos(nx) + i \sin(nx)) &= \cos x \cos nx + i \cos x \sin nx + i \sin x \cos nx + i^2 \sin x \sin nx \\
&= \cos x \cos nx + i \cos x \sin nx + i \sin x \cos nx - \sin x \sin nx \\
&= \cos(x + nx) + i \sin(x + nx) \\
&= \cos((n+1)x) + i \sin((n+1)x)
\end{aligned}$$

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3) (2 points each) Let $F(x) = x - 6$ and $G(x) = x^2 + 8$. Compute the values of the following, simplifying as much as possible.

(a) $F(G(y)) = y^2 + 8 - 6 = y^2 + 2$

(b) $G(F(y)) = (x-6)^2 + 8 = x^{12} - 12x + 36 + 8 = x^2 - 12x + 44$

(c) $\frac{F(F(G(2)))}{F(\pi)} = \frac{(((4+8)-6)-6)}{\pi - 6} = 0$

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4) (2 points each) Solve these short problems involving sums and products. Show your work. You can assume all variables are integers.

(a) Rewrite $\sum_{k=0}^{n}(k+1)\pi^k$ as a sum whose index runs from 4 to $n+4$.

(b) What is the first term of $\sum_{j=1}^{p} j2^j$? Write a summation for everything but the first term.

(c) My homework partner claims I can rewrite $\sum_{k=1}^{n} k^2$ as $\left(\sum_{k=1}^{n} k\right)\left(\sum_{k=1}^{n} k\right)$. Is he right?

(d) Evaluate this sum:

$$\sum_{j=0}^{n}(4j^3 + 5)$$

(a) $\sum_{k=4}^{n+4}(k-3)\pi^{k-4}$

(b) The first term is 2; the rest of the sum is $\sum_{j=2}^{p} j2^j$.

(c) No, this is incorrect. Let n equal 3. Then the first summation is $1 + 4 + 9 = 14$, and the second summation is $(1+2+3)^2 = 36$.

(d) First, we can split the sum at the plus sign to get

$$\sum_{j=0}^{n} 4j^3 + \sum_{j=0}^{n} 5$$

The second term here simplifies to $5n+5$, since there are $n+1$ 5's added together. Note that, from Rosen (p. 157), $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$; we factor out the 4 from this sum to get it into the proper form. However, note that the sum starts at index 0, and Rosen's formula starts at 1, so we have to bring out the first term and reindex:

$$\begin{aligned} 0 + 4\sum_{j=1}^{n} j^3 + 5n + 5 &= 4\frac{n^2(n+1)^2}{4} + 5n + 5 \\ &= n^4 + 2n^3 + n^2 + 5n + 5 \end{aligned}$$

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5) (3 points each)

(a) Translate the following sentence into propositional logic:

If poison ivy grows around here and this plant has groups of three leaves, then I need to wash my hands.

(b) For which values of p, q, and r is the following logical expression true (give a succinct description of which combinations of input values work, rather than the whole truth table).

$$(\neg p \vee q) \wedge (q \rightarrow r) \wedge (\neg r \vee p)$$

(c) Show that these two expressions are not logically equivalent, by giving specific values for p, q, and r for which the two expressions produce different values.

$$(p \rightarrow q) \wedge r$$

$$p \rightarrow (q \wedge r)$$

(a) Let x = poison ivy grows around here, y = this plant has groups of three leaves, and z = I need to wash my hands. The statement is then

$$(x \wedge y) \rightarrow z$$

(b) p, q, and r must either be all true or all false.

(c) Let p and r be false, and q be true. Then the first expression is false, and the second is true.