CS 173, Spring 2009
Midterm 2, 8 April 2009

INSTRUCTIONS (read carefully)

- Fill in your name, netid, and discussion section time below. Also write your netid on the other pages (in case they get separated).

NAME:

NETID:

DISCUSSION DAY/TIME:

- There are 6 problems, on pages numbered 1 through 6. Make sure you have a complete exam. The point value of each problem is indicated next to the problem, and in the table below.
- You have 50 minutes to finish the exam.
- Points may be deducted for solutions which are correct but excessively complicated, hard to understand, or poorly explained.
- Turn in your exam at the front. Show your ID to the proctors.
- It is wise to skim all problems and point values first, to best plan your time.
- This is a closed book exam. No notes or electronic devices of any kind are allowed. Do all work in the space provided, using the backs of sheets if necessary. See the proctor if you need more paper.
- Please bring any apparent bugs to the attention of the proctors.
- After the midterm is over, discuss its contents with other students **only** after verifying that they have also taken the exam (e.g. they aren't about to take a makeup exam).

Problem	1	2	3	4	5	6	total
Possible	10	8	10	8	8	6	50
Score							

Problem 1: Short answer (10 points)

The following require only short answers. Justification/work is not required, but may increase partial credit if your short answer is wrong.

(a) Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = \lfloor \frac{n}{3} \rfloor$. Is f one-to-one?

(b) In a balanced binary tree of height h, is it true that all the leaves are at level h?

(c) Is it true that $5n^3 + 17n$ is $\Omega(3^n)$?

(d) Snape's "Unfriendly Algorithms for Wizards" textbook claims the running time of merge sort is $O(n^4)$. Is this claim correct?

(e) Suppose that $h : \mathbb{Z} \to \mathbb{Z}$ is defined by $h(n) = n^2 - 2$. Give a counter-example that shows that h is not onto, stating extremely briefly why it is a counter-example.

Problem 2: Short answer (8 points)

(a) Recall that a **full *m-ary* tree** is a tree in which every node has either 0 or m children. If a full m-ary tree T has i internal vertices, write down an expression for the number of leaves in T. The only variables in the expression should be m and i. Briefly (in one or two sentences) explain how you arrived at your answer. (*Hint: you can use the formula you saw in lecture for the total number of vertices in a full m-ary tree.*)

(b) Suppose that f and g are functions whose input and output values are positive real numbers. Define precisely what it means for $f(x)$ to be $O(g(x))$. (Your definition cannot use the definition of some closely-related concept such as Ω.)

Problem 3: Recurrences (10 points)

(a) Solve the following recurrence using unrolling. Specifically, show at least two steps of unrolling, a summation whose value is equal to $T(n)$, and finally a closed-form expression (i.e. containing no recursion or summations) equal to $T(n)$.

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 2T(n-1) + 3 \end{aligned}$$

(b) Give a closed-form for the summation $\sum_{k=2}^{n} \left(\frac{1}{2}\right)^k$

Problem 4: Induction (8 points)

Use induction on n to prove the following fact:

For any integer $n > 4$, $n^2 < 2^n$.

Base case(s):

Inductive hypothesis:

Rest of the inductive step:

Problem 5: Algorithms (8 points)

In statistics, the *mode* is the value that occurs the most frequently in a data set. The following algorithm takes as input an arbitrary list of n integers a_1,...,a_n and attempts to find the mode of the list. The output is returned in the variable $modeValue$.

procedure FindMode(a_1,..., a_n)
$modeValue := a_1$
$modeCount := 1$
for $i := 1$ to $n - 1$
begin
  $count := 1$
  for $j := i + 1$ to n
  begin
    if $(a_i = a_j)$ **then**
      $count := count + 1$
  end
  if $(count > modeCount)$ **then**
  begin
    $modeValue := a_i$
    $modeCount := count$
  end
end
return $modeValue$

(a) Does the algorithm work correctly if the input list is $3, 3, 3, 1, 1$? What does this algorithm report as the mode?

(b) Give a big-theta bound on the number of equality tests (i.e. in the line **if** $(a_i = a_j)$ **then**) performed by this algorithm in the worst-case. Briefly explain how you derived your answer.

Problem 6: Recursive definition (6 points)

Here is a recursive definition of a set S, which contains pairs of integers.

1. $(4, 2)$ and $(4, 1)$ are in S.
2. If (x, y) is in S, then $(x + 1, y - 1)$ is in S.
3. If (x, y) is in S, then $(x - 1, y + 1)$ is in S.

Give a non-recursive definition for the set S. Explain briefly why your answer is correct.