CS 173, Spring 2009
Final Exam, 12 May 2009

Fill in your name, netid, and discussion section time below. Also write your netid on the other pages (in case they get separated).

NAME:

NETID:

DISCUSSION DAY/TIME:

Problem	1	2	3	4	5	6
Possible	8	11	8	9	10	10
Score						
Problem	7	8	9	10	11	
Possible	8	8	8	10	10	
Score						

Total out of 100 points

INSTRUCTIONS (read carefully)

- There are 11 problems, each on a single page. Make sure you have a complete exam.
- The point value of each problem is indicated next to the problem.
- It is wise to skim all problems and point values first, to best plan your time.
- Points may be deducted for solutions which are correct but excessively complicated, hard to understand, or poorly explained.
- Brief explanations and/or showing work may increase partial credit for buggy answers.
- We expect most people to finish the exam in 2 hours, but you can take up to the full 3 hours.
- Turn in your exam at the front. Show your ID to the proctors.
- This is a closed book exam. No notes or electronic devices of any kind are allowed.
- Except where explicitly indicated, it isn't necessary to simplify or calculate out complex constant expressions such as $(0.7)^3(0.3)^5$, $\frac{0.15}{3.75}$, $7!$, $\binom{10}{7}$, and the like.
- Do all work in the space provided, using the backs of sheets if necessary. See the proctor if you need more paper.
- Please bring any apparent bugs to the attention of the proctors.
- After the exam is over, discuss its contents with other students **only** after verifying that they have also taken the exam (e.g. they aren't about to take a conflict exam).

Problem 1: Multiple choice (8 points)

Check the appropriate box for each statement. (One box per statement.) If you change your answer, make it very clear when you've meant to uncheck a box.

In 1976, researchers at U.C. Berkeley proved that any (undirected) planar graph can be colored with four colors, using a computer program to check 1936 special cases. **True** ☐ **False** ☐

The function $f(n) = 2^n$ is $\Omega(n!)$. **True** ☐ **False** ☐

No subset of the real numbers is countable. **True** ☐ **False** ☐

The empty set is a member of $\mathbb{P}(\{a, b\}) \times \mathbb{P}(\{p, q\})$. **True** ☐ **False** ☐

If the graph of a relation contains no arrows at all, then the relation is symmetric. **True** ☐ **False** ☐

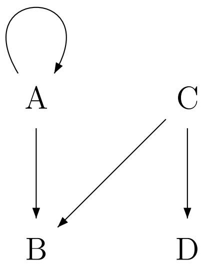
Zero is a multiple of 7. **True** ☐ **False** ☐

The worst-case running time of merge sort is $O(n)$ ☐ $O(n \log n)$ ☐ $O(n^2)$ ☐ $O(2^n)$ ☐

The worst-case running time of insertion sort is $O(n)$ ☐ $O(n \log n)$ ☐ $O(n^2)$ ☐ $O(2^n)$ ☐

Problem 2: Checkbox (11 points)

Check all boxes which correctly characterize each relation or function, leaving the other boxes blank. (If you change your answer, make it very clear when you've meant to uncheck a box.)



Reflexive: ☐ **Irreflexive**: ☐

Symmetric: ☐ **Antisymmetric**: ☐

Transitive: ☐

$f : \mathbb{R} \to \mathbb{R}$
such that
$f(x) = 3x^2 + 37$

One-to-one: ☐ **Onto**: ☐

$f : \mathbb{N} \to \mathbb{N}$
such that
$f(n) = 3n^3 - 7n^2 + 37$

$O(n^3)$: ☐ $O(2^n)$: ☐

$\Omega(n^3)$: ☐ $\Omega(2^n)$: ☐

Problem 3: Counting (8 points)

(a) (5 points) Imagine a two-dimensional grid consisting of 20 points along the x-axis and 10 points along the y-axis. Suppose the origin (0,0) is in the bottom-left corner and the point (20,10) is the top-right corner. A path on the grid consists of a series of moves in which each move is either one unit to the right or one unit up. Diagonal moves are not allowed. How many different ways are there to construct a path starting at (0,0) and ending at (20,10)?

(b) (3 points) If set $A = \{a, b, c, d\}$ and set $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ how many possible functions $f : A \rightarrow B$ are there from A to B?

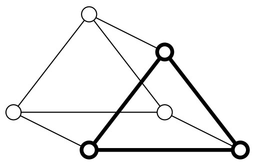
Problem 4: Probability (9 points)

(a) (4 points) Suppose the Chicago White Sox play the Cincinnati Reds 8 times. In each game, the probability of the White Sox winning is 0.75. What is the probability of the White Sox winning exactly 3 of the 8 games?

(b) (5 points) Imagine two integers a and b are randomly generated with uniform probability and that each integer will have a value of 1,2, or 3. Suppose we define a random variable $X = \text{maximum}(a, b)$. For example, if $a = 2$ and $b = 1$ then $X = 2$. What is the *expected value* of X? *Show your work.*

Problem 5: Short answer (10 points)

(a) (4 points) Suppose we define a graph G_1 to be a cycle containing 3 vertices. We define G_n to be the result taking two copies of G_{n-1} and adding edges between the corresponding pairs of vertices. For example, G_2 is shown below, with one copy of G_1 shown in bold. Let T_n be the *number of edges* in G_n, e.g. $T_2 = 9$. Write down a recurrence relation (with a base case) for T_n.



(b) (3 points) Define an equivalence relation $\sim$ on $\mathbb{Z}^2$ by $(m, n) \sim (p, q)$ if and only if $mn = pq$. List three elements of $[(4, 3)]$.

(c) (3 points) Define the relation R on Z^2 such that $(x, y)R(p, q)$ if and only if $xp + yq = 0$. Give a concrete counterexample that shows that R is *not* transitive, briefly explaining why it is a counter-example.

Problem 6: Proofs and definitions (10 points)

(a) (4 points) Suppose that A is a set and P is a set of subsets of A. Define what it means for P to be a partition of A.

(b) (6 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = 2n + 13$. Prove that f is one-to-one. Do this directly from the definition of one-to-one. Do not use the fact that it's increasing (or similar reasoning involving its slope/derivative).

Problem 7: Unrolling (8 points)

In this question, you will find a closed-form solution for the following recurrence relation: $T(n) = 2T(n-1) + 2$ with $T(0) = 1$

(a) (3 points) Start by writing down the result of *unrolling* the recurrence twice to get an expression for $T(n)$ in terms of $T(n-3)$

(b) (3 points) Now, write down an expression for $T(n)$ using summation notation.

(c) (2 points) What is a closed-form solution for $T(n)$?

Problem 8: Trees and graphs (8 points)

On a connected simple graph G we can measure the distance between two distinct vertices a and b as the number of edges on the shortest path between them. (The distance from a vertex to itself is zero.) The *diameter* of a graph G is the maximum distance between any two distinct vertices in G.

Suppose that our graph G is a balanced binary tree with v vertices in which every level of the tree is completely filled. That is, all leaves are at the same height, which we'll call h.

(a) (3 points) Express the number of vertices v as a function of the tree height h.

(b) (2 points) What is the diameter d of the tree as a function of its height h?

(c) (3 points) Using big-theta notation, express the diameter d of the tree as a function of the number of vertices v.

Problem 9: Algorithm analysis (8 points)

The following procedure takes as input a set of n unordered positive integers $a_1, a_2, ...a_n$ and computes a single integer m that it returns as the result. The operation $|(a_i - a_j)|$ computes the absolute value of $a_i - a_j$. Here is the procedure:

```
1  procedure ComputeIt(a_1,..., a_n)
2  m := 0
3  for i := 1 to n − 1
4  begin
5      for j := i + 1 to n
6      begin
7          if |(a_i − a_j)| > m then
8              m := |(a_i − a_j)|
9      end
10 end
11 return m
```

(a) (3 points) What value does the algorithm return if the input list is $4, 13, 20, 5, 8, 10$

(b) (3 points) Give a big-theta bound on the number of times line 7 is executed.

(c) (2 points) In a single sentence, describe what the procedure computes (in other words, describe what m represents).

Problem 10: Writing a proof (10 points)

Recall the definition of congruence mod m

$p \equiv q \pmod{m}$ if and only if there is an integer k such that $p - q = km$.

Using this definition plus high-school algebra, show that

If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.

Use only this definition and not other facts about modular arithmetic you may remember from class. Be sure to use good mathematical style, with the steps in logical order and appropriate justifications.

Problem 11: Induction (10 points)

Suppose that the function g is defined on all positive integers by:

$$g(n) = \sum_{k=1}^{n} k(k+2)(k+4)$$

Fill in the boxes below to create an inductive proof that $g(n) = \frac{1}{4}n(n+1)(n+4)(n+5)$, for every integer $n \geq 1$.

Base case or cases:

Inductive hypothesis:

Main work of the inductive step:

Conclusion of your inductive step:

Proof conclusion: so $g(n) = \frac{1}{4}n(n+1)(n+4)(n+5)$, for every integer $n \geq 1$.