



CS 173: Discrete Structures

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Office Hour: Wed. 1-2, 2215 SC

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Announcements

- Midterm Exam next Wed. 11/5: In Class, Please Attend
 - List of topics is up on the webpage
 - Quiz 2 Feedback
 - 170 people took the quiz
 - range of scores: [9,25] out of 25
 - mean 18.6, median 19
 - More importantly, use the quiz results to help you study
 - Know functions (1-1, onto, composition), O , Ω , θ
- Just to clear up some confusion:

f and g are functions on reals or integers (doesn't matter)
 $f(x) = O(g(x))$ if there exist real constants m, k such that
 $|f(x)| \leq m|g(x)|$ for $x \geq k$





Today...probability

- Sections 6.1 and 6.2 of the book
- Discrete probability
- Why should you study probability? Here's a few reasons:
 - Average case analysis of algorithms (CS 225)
 - Randomized algorithms and data structures (skip list)
 - Ability to excel in your fantasy baseball league
 - Network communication protocols
 - Spam filters
 - Computer graphics (monte carlo light transport)
 - Computational simulation in general...
 - Detecting plagiarism
 - Gambling (or "investing"), which is how it all started...





Great moments in Probability throughout history

- 1654: French mathematicians Pascal and Fermat correspond to solve this problem: two players want to finish a dice game early and, given the current circumstances of the game, want to divide the stakes fairly based on the chance each has of winning the game from that point. Could you solve "The Problem of Points"?
- 1867: Dostoyevsky publishes one of his best and shortest novels, *The Gambler*, in order to pay off his own gambling debts. He played roulette using a martingale strategy (double your bet after every loss). Is this a good strategy?





Great moments in Probability throughout history

- 1962: Edward Thorp, American mathematician, publishes "Beat the Dealer", a book describing a winning Blackjack strategy. Thorp analyzed blackjack using Fortran programs he wrote on an IBM 704.
 - "Thorp became one of the very few applied mathematicians who risked physical harm in verifying a computer simulation." - Wikipedia
 - In 1974 founded Princeton/Newport Partners, the first(?) hedge fund, and proceeded to make a small fortune while helping establish the field of computational finance.





Scary picture and story of the day





Probability



Which is more likely:

- a) Rolling an 8 when 2 dice are rolled?
- b) Rolling an 8 when 3 dice are rolled?
- c) No clue.





Probability



Sample Space S : set of possible outcomes of an experiment
Event E : subset of the sample space

Assuming each outcome is equally likely $\Pr(E) = |E|/|S|$

Note that $\Pr(E) + \Pr(\bar{E}) = 1$

And: $\Pr(E1) \cup \Pr(E2) = \Pr(E1) + \Pr(E2) - \Pr(E1 \cap E2)$

D1
1
1

D2
6
5

...





Probability



Let s be outcome in the Sample Space S

A function $p(s)$ that assigns a probability to each $s \in S$ is called a Probability Distribution

Note: $0 \leq p(s) \leq 1$ and $\sum p(s_i) = 1$

We have been assuming that S is finite

It could also be a countably infinite set

It cannot have real-valued outcomes

(that requires continuous mathematics)





Probability



What is the probability of a total of 8 when 2 dice are rolled?

What is the size of the sample space? 36

How many rolls satisfy our condition of interest? 5

So the probability is $5/36$.

$(2, 6)$, $(6, 2)$
 $(5, 3)$ $(3, 5)$
 $(4, 4)$





Probability



What is the probability of a total of 8 when 3 dice are rolled?

What is the size of the sample space? 216

How many rolls satisfy our condition of interest? $C(7,2)$



So the probability is $21/216$.





Lotteries

Pick 4: Win with the correct 4 digits in order

$$\Pr(4\text{of}4) = \frac{1}{10^4} = 0.0001$$

Also a prize for 3 of 4 correct digits

$$\Pr(3\text{of}4) = \frac{C(4,1) \cdot 9}{10^4} = \frac{36}{10^4} = 0.0036$$

Sort of Powerball: Choose 6 numbers correctly from the first 40 integers ← order not important

$$\Pr(6) = \frac{1}{C(40,6)} = \frac{1}{3,838,380} \approx 0.00000026$$





Poker

Deck has 52 cards, 4 "suits", each suit has 13 "ranks"

A poker hand consists of 5 cards

"4 of a kind" means a hand with 4 cards of the same rank

Order is unimportant.....

$\text{Pr}(4\text{ofakind}) = (\# \text{ ways to choose rank } \times$
 $\# \text{ ways to pick the 4 } \times$
 $\# \text{ ways to pick the final card}) / (\text{total hands})$

$$\begin{aligned}\text{Pr}(4\text{ofakind}) &= C(13,1)C(4,4)C(48,1)/C(52,5) \\ &= 13(1)(48)/(2598960) = \text{about } 0.00024\end{aligned}$$





Bitstrings

A string of 10 bits is randomly generated
What is the probability of it having at least one "0"?

$$\Pr(\text{1 or more}) = 1 - \Pr(\text{none})$$

$$\Pr = 1 - \frac{1}{2^{10}} = \frac{1023}{1024}$$





One last example..

Pick an a positive integer not bigger than 100.
What is the probability it is divisible by 2 or 5?

$$P_r = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{60}{100}$$

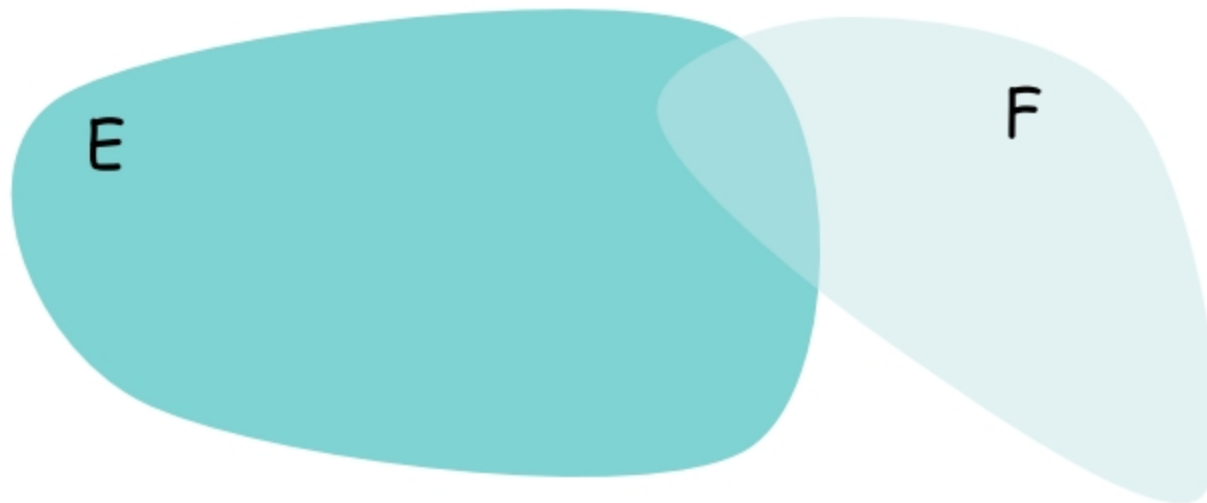




Conditional Probability

Let E and F be events with $\Pr(F) > 0$. The conditional probability of E given F , denoted by $\Pr(E|F)$ is defined to be:

$$\Pr(E|F) = \Pr(E \cap F) / \Pr(F).$$





Conditional Probability

$$\Pr(E|F) = \Pr(E \cap F) / \Pr(F).$$

A bit string of length 4 is generated at random so that each of the 16 bit strings is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

$$\Pr(F) = 1/2$$

$$\Pr(E \cap F)? \quad \underbrace{0000}, \underbrace{0001}, \underbrace{0010}, \underbrace{0011}, \underbrace{0100}$$

$$\Pr(E \cap F) = 5/16$$

$$\Pr(E|F) = 5/8$$





Independence

The events E and F are *independent* if and only if $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$.

Let E be the event that a family of n children has children of both sexes.
Let F be the event that a family of n children has at most one boy.

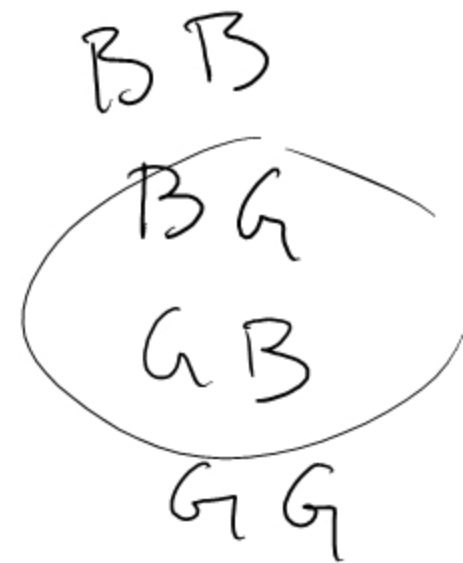
Are E and F independent if

$n = 2$? **No**

$$\Pr(E) = \frac{1}{2}$$
$$\Pr(F) = \frac{3}{4}$$

$$\Pr(E \cap F) = \frac{1}{2}$$

$$\frac{1}{2} \neq \frac{1}{2} \times \frac{3}{4}$$





Independence

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Are E and F independent if

$n = 3?$ Yes





Independence

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Let E be the event that a family of n children has children of both sexes.

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Are E and F independent if

$n = 2?$ No

$n = 4?$ No

$n = 3?$ Yes

$n = 5?$ No





Bernoulli Trials

A coin is tossed 8 times. What is the probability of exactly 3 heads in the 8 tosses?

THHTTHTT is a tossing sequence...

How many ways of choosing 3 positions for the heads?

$$C(8,3)$$

What is the probability of a particular sequence?

$$.5^8$$

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p , is

$$C(n,k)p^k(1-p)^{n-k}$$





Bernoulli Trials

A game of Jewel Quest is played 5 times. You clear the board 70% of the time. What is the probability that you win a majority of the 5 games?

Sanity check: What is the probability the the result is WWLLW? $.7^3.3^2$ Assumes independent trials

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p , is $C(n,k)p^k(1-p)^{n-k}$
 $C(5,3)0.7^30.3^2 + C(5,4)0.7^40.3^1 + C(5,5)0.7^50.3^0$





Birthdays

How many people have to be in a room to assure that the probability that at least two of them have the same birthday is greater than $1/2$?

- a) 23
- b) 183
- c) 365
- d) 730

Let p_n be the probability that no people share a birthday among n people in a room.

Then $1 - p_n$ is the probability that 2 or more share a birthday.

We want the smallest n so that $1 - p_n > 1/2$.





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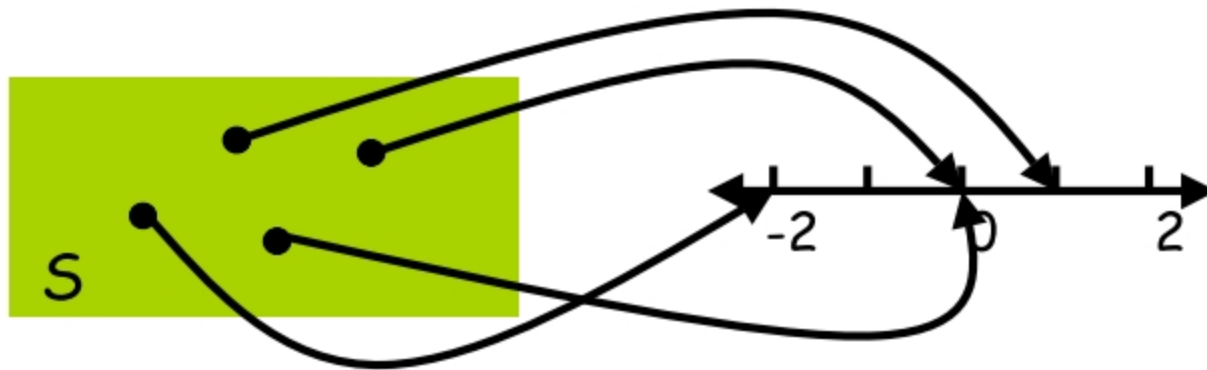
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Random Variables

For a given sample space S , a *random variable* is any real valued function on S .



Suppose our experiment is a roll of 2 dice. S is set of pairs.

- X = sum of two dice. $X((2,3)) = 5$
- Y = difference between two dice. $Y((2,3)) = 1$
- Z = max of two dice. $Z((2,3)) = 3$





Random Variables

A probability distribution on a r.v. X is just an allocation of the total probability, 1, over the possible values of X .

How many movies have you watched in the last week?

- a) 0
- b) 1
- c) 2
- d) 3
- e) 4

Picture gives a probability distribution!

The chart gives likelihood that a randomly selected student watched each of the particular numbers of movies.





Random Variables

Example: Do you ever play the game Racko?

Suppose we are playing a game with cards labeled 1 to 20, and we draw 3 cards. We bet that the maximum card has value 17 or greater. What's the probability we win the bet?

Let r.v. X denote the maximum card value. The possible values for X are 3, 4, 5, ..., 20.

i	3	4	5	6	7	8	9	...	20
$\Pr(X = i)$?	?	?	?	?	?	?		?



Filling in this box would be a pain. We look for a general formula.



Random Variables

X is value of the highest card among the 3 selected. 20 cards are labeled 1 through 20.

We want $\Pr(X = i)$, $i = 3, \dots, 20$.

Denominator first: How many ways are there to select the 3 cards? $C(20,3)$

How many choices are there that result in a max card whose value is i ? $C(i-1,2)$

$\Pr(X = i) = C(i-1, 2) / C(20,3)$ These are the table values.

We win the bet if the max card, X is 17 or greater. What's the probability we win?

$$\Pr(X = 17) + \Pr(X = 18) + \Pr(X = 19) + \Pr(X = 20)$$

≈ 0.51

- a) 20
- b) 6840
- c) 60
- d) 1140
- e) I'm not telling.





Expected Value

Let X be a discrete r.v. with set of possible values D . The *expected value* of X is:

$$E(X) = \sum_{x \in D} x \cdot \Pr(X = x)$$

Measure of central tendency.

Let X denote your score on the coming midterm.
Suppose I assign scores according to the following distribution:

i	55	65	80	90
$\Pr(X=i)$	0.1	0.3	0.4	0.2

$$\begin{aligned} \text{Then } E(X) &= (55)(0.1) + \\ &65(0.3) + (80)(0.4) + \\ &90(0.2) = 75 \end{aligned}$$





Expected Value

Let X be a binomial r.v. with parameters n and p .
That is, X is the number of "successes" on n trials
where each trial has probability of success p .

What is $E(X)$?

Defn of Binomial
Distribution.

First we need $\Pr(X = k) = C(n, k) p^k (1-p)^{n-k}$

$$E(X) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$





Expected Value

$$\begin{aligned} E(X) &= \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n \frac{k \cdot n!}{(n-k)!k!} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k} \\ &= np \sum_{k=0}^n \frac{n-1!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k} \\ &= np \sum_{k=0}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \\ &= np [p + (1-p)]^{n-1} = np \end{aligned}$$





Expected Value

Let $X_i, i= 1,2,\dots,n$, be a sequence of random variables, and suppose we are interested in their sum. The sum is a random variable itself with expectation given by:

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i]$$

The proof of this is inductive and algebraic. You can find it in your book on page 382.





Expected Value

Suppose you all (n) put your cell phones in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back?

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i]$$

Define for $i = 1, \dots, n$, a random variable:
$$X_i = \begin{cases} 1 & \text{if student } i \text{ gets the right phone,} \\ 0 & \text{otherwise.} \end{cases}$$

k	0	1
$\Pr(X_i=k)$	$1 - (1/n)$	$1/n$

- $E[X_i] =$
- a) $1/n$
 - b) $1/2$
 - c) 1
 - d) No clue

$$E[X_i] = \Pr(X_i = 1)$$





Expected Value

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Define for $i = 1, \dots, n$, a random variable:

$$X_i = \begin{cases} 1 & \text{if student } i \text{ gets the right phone,} \\ 0 & \text{otherwise.} \end{cases}$$

Define r.v. $X = X_1 + X_2 + \dots + X_n$, and we want $E[X]$.

$$\begin{aligned} E[X] &= E[X_1 + X_2 + \dots + X_n] \\ &= E[X_1] + E[X_2] + \dots + E[X_n] = 1/n + 1/n + \dots + 1/n = 1 \end{aligned}$$





Expected Value

Suppose there are N couples at a party, and suppose m people get sleepy and leave. What is the expected number of couples left?

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i]$$

Define for $i = 1, \dots, N$, a random variable:

$$X_i = \begin{cases} 1 & \text{if couple } i \text{ remains,} \\ 0 & \text{otherwise.} \end{cases}$$

Define r.v. $X = X_1 + X_2 + \dots + X_n$, and we want $E[X]$.

$$E[X] = E[X_1 + X_2 + \dots + X_n] \\ = E[X_1] + E[X_2] + \dots + E[X_n]$$

So what do we know about X_i ?





Expected Value

Suppose there are N couples at a party, and suppose m people get sleepy and leave. What is the expected number of couples left?

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i]$$

Define for $i = 1, \dots, N$, a random variable:

$$X_i = \begin{cases} 1 & \text{if couple } i \text{ remains,} \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X_i] = \Pr(X_i = 1) \quad \begin{array}{l} (\# \text{ of ways of choosing } m \text{ from everyone else}) \\ / (\# \text{ of ways of choosing } m \text{ from all}) \end{array}$$

$$= \frac{\binom{2N-2}{m}}{\binom{2N}{m}}$$

$$E[X_1] + E[X_2] + \dots + E[X_n]$$

$$= n \times E[X_1] = (2N-m)(2N-m-1)/2(2N-1)$$

