Binomial coefficients

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This lecture covers the rest of section 5.4 of Rosen, plus some material from section 5.5.

1 Announcements

Exam a week from Wednesday (Nov 5th) in class. Tell Margaret ASAP if you have a conflict. Exam skills list should be posted sometime today. We expect to return the quizzes in class Friday.

2 Another combinatorial proof

At the end of last class, we saw the following identity:

Claim 1 *(Pascal's identity)* $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

There are (at least) two ways to prove this. We can prove it directly from the factorial equations:

Proof:

$$
\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} \cdot \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(k-1)!((n+1)-k)!} \cdot \frac{n!}{k!(n-k)!} \\
&= \frac{kn!}{k!((n+1)-k)!} \cdot \frac{(n+1-k)n!}{k!((n+1)-k)!} \\
&= \frac{kn! + (n+1-k)n!}{k!((n+1)-k)!} \\
&= \frac{(n+1)n!}{k!((n+1)-k)!} \\
&= \frac{(n+1)!}{k!((n+1)-k)!} \\
&= \binom{n+1}{k}
\end{aligned}
$$

Alternatively, we can use a combinatorial proof:

Proof: Suppose that S is a set with $n+1$ elements. $\binom{n+1}{k}$ is the number of k-element subsets of S.

Pick some element $a \in S$. We can divide the k-element subsets of S into two groups: those that contain a and those that don't.

The k-element subsets of S that don't contain a are the same as the k-element subsets of $S - \{a\}$, i.e. the k-element subsets of a set with n elements. This is $\binom{n}{k}$.

The k-element subsets of S that do contain a can be formed by adding a onto each $k-1$-element subsets of $S - \{a\}$. There are $\binom{n}{k-1}$ of these.

So there are $\binom{n}{k-1} + \binom{n}{k}$ k-element subsets of S. That is $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Both methods of proof work fine. The combinatorial proofs are worth knowing about because they are sometimes easier to read/write.

3 Vandermonde's Identity

Here's another useful identity:

Claim 2 *(Vandermonde's Identity)* $\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$

We can prove this using a combinatorial argument, as follows:

> Proof: Suppose that we have a set A with n elements and a set B with m elements, where A and B don't overlap. If we want to choose r elements from $A \cup B$, we will have to pick some number k from A and then the remaining $r - k$ from B.
>
> So, we need to walk through all possible values of k. For each value of k, we pick k elements from A. This can be done in $\binom{n}{k}$ ways. We then pick $r - k$ elements from B, which can be done in $\binom{m}{r-k}$ ways. So, using the sum and product rule, the total number of choices is
>
> $$\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

4 Binomial Theorem

A *binomial* is a sum of two terms, e.g. $(x + y)$. The *binomial theorem* shows how to raise a binomial to any integer power. Specifically

Claim 3 *(Binomial Theorem) Let x and y be variables and let n be any natural number. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Because of this application, the values $\binom{n}{k}$ are sometimes called **binomial coefficients**.

> Proof: If we were to expand the product $(x+y)^n$, each term is the product of n variables, some x's and the rest y's. For example, if $n = 6$, one term is $yxyyxx$. So each term is an ordered list of x's and y's.
>
> When we collect up terms, we group together the lists that have the same number of x's. To find the coefficient for $x^{n-k}y^k$, we need to count how many ways we can make a list of n elements that contains k y's and $n-k$ x's. This amounts to picking a subset of k elements from a set of n positions in the list. In other words, there are $\binom{n}{k}$ such terms.

5 Corollaries of the Binomial Theorem

Suppose that we set $x = 1$ in the Binomial Theorem. Then we have

$$(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} y^k$$

So we have the following corollary.

Claim 4 *For any variable y and any natural number n,* $(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k$

A corollary is a result that is very easy to prove, once you've proved some theorem (which often had a hard proof).

We could also state this as:

Claim 5 *For any variable y and any natural number n, $\sum_{k=0}^{n} \binom{n}{k} y^k = (1+y)^n$.*

If we plug some specific values of y into this formula, we get some nice results about sums of binomial coefficients. For example, if $y = 1$, then we have $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

One way to understand this equation is that we are counting all subsets of a set S that contains n elements. 2^n is the total number of subsets. The summation on the left considers each possible size of subset (k). For each size k, it computes the number of subsets of size k.

If $y = -1$, then we have

$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0.$

If $y = 2$, then we have

$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n.$

We don't expect that you'll remember all these random identities involving binomial coefficients. Rather, we're hoping that you remember the important named ones, and that you could figure out how to rederive the others (given some time to fiddle around).

6 Permutations with identical objects

Life gets more interesting when our problem involves multiple copies of the same (or indistinguishable) objects. I'm not going to walk through all the possibilities here, because they get ugly fast. I'll just show a couple cases where the analysis is still fairly nice.

First, suppose I have a list of objects such as $L = (c, o, l, l, e, g, e)$ that contains some duplicates. How many ways can I put the elements of L into different orders?

If we ignored the fact that some objects are duplicates, we would calculate

7! permutations of this list. However, this is double-counting some possibilities, because it doesn't matter what order we put the duplicates in. So we need to divide out by the number of ways we can permute the duplicates. In this case, we have 2! ways to permute the l's and 2! ways to permute the e's. So the true number of orderings of L is $\frac{7!}{2!2!}$.

Similarly, the number of reorderings of $J = (a, p, p, l, e, t, r, e, e, s)$ is $\frac{10!}{2!3!}$.

In general, suppose we have n objects, where n_1 are of type 1, n_2 are of type 2, and so forth through n_k are of type k. Then the number of ways to order our list of objects is $\frac{n!}{n_1!n_2!...n_k!}$.

7 Combinations with repetition

Suppose I have a set S and I want to select a group of objects of the types listed in S, but I'm allowed to pick more than one of each type of object. For example, suppose I want to pick 6 plants for my garden and the set of available plants is $S = \{\text{thyme}, \text{oregano}, \text{mint}\}$. The garden store can supply as many as I want of any type of plant. I could pick 3 thyme and 3 mint. Or I could pick 2 thyme, 1 oregano, and 3 mint.

There's a clever way to count the possibilities here. Let's draw a picture of a selection as follows. Let's put down a star for each thyme selected, then a separator #, then a star for each oregano selected, another #, then a star for each mint So 2 thyme, 1 oregano, and 3 mint looks like

** # * # ***

And 3 thyme and 3 mint looks like

*** ## ***

To count these pictures, we need to count the number of ways to arrange 6 stars and two #'s. That is, we have 8 positions and need to choose 2 to fill with #'s. In other words, $\binom{8}{2}$.

In general, suppose we are picking a group of k objects (with possible duplicates) from a list of n types. Then our picture will contain k stars and $n-1$ #'s. So we have $k+n-1$ positions in the picture and need to choose $n-1$ positions to contain the #'s. So the number of possible pictures is $\begin{pmatrix} k+n-1 \\ n-1 \end{pmatrix}$.

Notice that this is equal to $\begin{pmatrix} k+n-1 \\ k \end{pmatrix}$ because we have an identity that says so (see last lecture). We could have done our counting by picking a subset of k positions in the diagram that we would fill with stars (and then the rest of the positions will get the #'s).

If wanted to pick 20 plants and there were five types available, I would have $\begin{pmatrix} 24 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 20 \end{pmatrix}$ options for how to make my selection. $\begin{pmatrix} 24 \\ 4 \end{pmatrix} = \frac{24 \cdot 23 \cdot 22 \cdot 21}{4 \cdot 3 \cdot 3} = 23 \cdot 22 \cdot 21$.