noted that the boundary condition (9.1) takes on different forms as we traverse the outside and inside of a rectangular region with a hole. The various forms for

$$-\left(D_x\frac{\partial\phi}{\partial x}\cos\theta+D_y\frac{\partial\phi}{\partial y}\sin\theta\right) \tag{9.15}$$

are given in Figure 9.2. The sign associated with M and S must be determined relative to the information contained in Figure 9.2. This determination is illustrated in the application chapters that follow. Remember that θ is the angle from the x-axis to the outward normal.

9.2 EVALUATION OF THE ELEMENT INTEGRALS

The integrals in (9.13) and (9.14) are valid for any two-dimensional element and can be evaluated once the element shape functions are known. We shall evaluate these for the linear triangular and bilinear rectangular elements that were studied in Chapter 5. We shall start with (9.14) and the rectangular element because this combination is the easiest to evaluate.

Assuming that S is specified over side ij and that the element has a unit thickness gives

$$\int_{\Gamma_{bc}} S[N]^T d\Gamma=\int_{-b}^{b} S\begin{Bmatrix}N_i\\N_j\\N_k\\N_m\end{Bmatrix} dq \tag{9.16}$$

where the shape functions are those defined by (5.19) for the qr-coordinate system. Note, however, that $N_k=N_m=0$ along side ij. Substituting for the nonzero shape functions and noting that $r=-a$,

$$\{f_S^{(e)}\}=\int_{-b}^{b}\frac{S}{2b}\begin{Bmatrix}(b-q)\\(b+q)\\0\\0\end{Bmatrix}dq=\frac{SL_{ij}}{2}\begin{Bmatrix}1\\1\\0\\0\end{Bmatrix} \tag{9.17}$$

The quantity S is multiplied by the length of side ij, L_{ij}, which is $2b$ and divided equally between the two nodes on side ij.

There are three other evaluations for the surface integral, one for each of the remaining three sides. It is left as an exercise to show that the other results for $\{f_S^{(e)}\}$ are

$$\{f_S^{(e)}\}=\frac{SL_{jk}}{2}\begin{Bmatrix}0\\1\\1\\0\end{Bmatrix},\qquad \frac{SL_{km}}{2}\begin{Bmatrix}0\\0\\1\\1\end{Bmatrix},\qquad\text{and}\qquad \frac{SL_{im}}{2}\begin{Bmatrix}1\\0\\0\\1\end{Bmatrix} \tag{9.18}$$

for sides jk, km, and im, respectively. If S is specified on more than one side of an element, the values for $\{f_S^{(e)}\}$ for the appropriate sides are added together.

The evaluation of (9.14) for the triangular element gives results that are very similar to those in (9.17) and (9.18). The results are

$$\{f_S^{(e)}\}=\frac{SL_{ij}}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix},\qquad \frac{SL_{jk}}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix},\qquad\text{and}\qquad \frac{SL_{ik}}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix} \tag{9.19}$$

for sides ij, jk, and ik, respectively. The quantities L_{ij}, L_{jk}, and L_{ik} are the lengths of the respective sides. They are not the area coordinates. The area coordinates have numerical subscripts.

The first result of (9.19) is obtained as follows. Given side ij,

$$\{f_S^{(e)}\}=\int_{\Gamma_{bc}} S[N]^T d\Gamma=L_{ij}\int_0^1 S\begin{Bmatrix}N_i\\N_j\\N_k\end{Bmatrix} d\ell_2 \tag{9.20}$$

Since N_k is zero along side ij,

$$\{f_S^{(e)}\}=L_{ij}\int_0^1 S\begin{Bmatrix}N_i\\N_j\\0\end{Bmatrix} d\ell_2=L_{ij}\int_0^1 S\begin{Bmatrix}\ell_1\\\ell_2\\0\end{Bmatrix} d\ell_2 \tag{9.21}$$

because the shape functions N_i and N_j reduce to

$$N_i=L_1=\ell_1 \qquad\text{and}\qquad N_j=L_2=\ell_2 \tag{9.22}$$

along side ij. This fact was discussed in Chapter 6. The integration is along a line; thus we can use the factorial formula (6.17), and the result in (9.19) follows immediately.

The integrals associated with $[k_M^{(e)}]$ are evaluated in a manner identical to those just discussed. The major difference is that there are more terms to consider. The integral in (9.13) expands into

$$[k_M^{(e)}]=\int_{\Gamma_{bc}} M\begin{bmatrix}N_i^2 & N_iN_j & N_iN_k & N_iN_m\\ N_iN_j & N_j^2 & N_jN_k & N_jN_m\\ N_iN_k & N_jN_k & N_k^2 & N_kN_m\\ N_iN_m & N_jN_m & N_kN_m & N_m^2\end{bmatrix} d\Gamma \tag{9.23}$$

for the rectangular element. If we assume that M is specified over side ij, then $N_k=N_m=0$ and (9.23) becomes

$$[k_M^{(e)}]=\int_{-b}^{b} M\begin{bmatrix}N_i^2 & N_iN_j & 0 & 0\\ N_iN_j & N_j^2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} dq \tag{9.24}$$

Evaluation of the individual coefficients after noting $r=-a$ gives

$$\int_{-b}^{b} N_i^2\,dq=\int_{-b}^{b}\frac{(b-q)^2}{4b^2}\,dq=\frac{2b}{3}=\frac{L_{ij}}{3} \tag{9.25}$$

$$\int_{-b}^{b} N_i N_j \, dq = \int_{-b}^{b} \frac{(b-q)(b+q)}{4b^2} \, dq = \frac{2b}{6} = \frac{L_{ij}}{6} \tag{9.26}$$

and

$$\int_{-b}^{b} N_j^2 \, dq = \int_{-b}^{b} \frac{(b+q)^2}{4b^2} \, dq = \frac{2b}{3} = \frac{L_{ij}}{3} \tag{9.27}$$

Using these results, we have

$$[k_M^{(e)}] = \frac{ML_{ij}}{6} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9.28}$$

There are three other results for $[k_M^{(e)}]$, one for each of the other sides. These results are

$$[k_M^{(e)}] = \frac{ML_{jk}}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9.29}$$

$$[k_M^{(e)}] = \frac{ML_{km}}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \tag{9.30}$$

and

$$[k_M^{(e)}] = \frac{ML_{im}}{6} \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \tag{9.31}$$

where L_{jk}, L_{km}, and L_{im} are the lengths of the respective sides.

The evaluation of (9.13) for the triangular element leads to

$$[k_M^{(e)}] = \frac{ML_{ij}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9.32}$$

$$[k_M^{(e)}] = \frac{ML_{jk}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{9.33}$$

and

$$[k_M^{(e)}] = \frac{ML_{ik}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \tag{9.34}$$

9.3 POINT SOURCES AND SINKS

An important physical situation is the concept of a point source or sink. A source or sink is said to exist whenever Q occurs over a very small area. Examples of line sources include steam and/or hot water pipes within the earth and conducting electrical wires embedded within a product. In each case, the cross-sectional area of the pipe or conductor is very small compared with the surrounding media. Point sinks occur in groundwater problems: They are pumps removing water from an aquifer.

Sources and sinks occur often enough in the real world to warrant our attention. Our discussion is structured around the two-dimensional element, but the procedure can be quickly modified to handle the one- or three-dimensional element.

Consider the triangular element in Figure 9.3 with a source Q^* located at (X_0, Y_0). Since the source is located at a point, Q is no longer constant throughout the element but is a function of x and y. Using unit impulse functions, $\delta(x-X_0)$ and $\delta(y-Y_0)$ (Kaplan, 1962), we can write

$$Q = Q^* \, \delta(x-X_0)\, \delta(y-Y_0) \tag{9.35}$$

The integral

$$\{f_Q^{(e)}\} = \int_A Q[N]^T \, dA \tag{9.36}$$

becomes

$$\{f_Q^{(e)}\} = Q^* \int_A \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} \delta(x-X_0)\, \delta(y-Y_0) \, dx \, dy \tag{9.37}$$

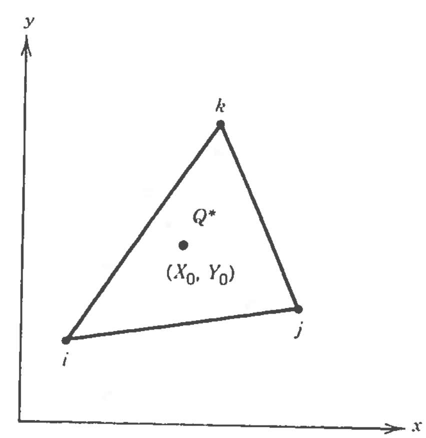

Figure 9.3. An element with a point source or sink.

The integral of a quantity multiplied by an impulse function, however, is equal to the quantity evaluated at X_0 and Y_0. Therefore,

$$\{f_Q^{(e)}\} = Q^* \begin{Bmatrix} N_i(X_0, Y_0) \\ N_j(X_0, Y_0) \\ N_k(X_0, Y_0) \end{Bmatrix} \tag{9.38}$$

The proportion of Q^* allocated to each node is based on the relative values of N_i, N_j, and N_k evaluated using the coordinates of the point source. Since the shape functions sum to one at every point within the element, we are not allocating more than Q^*.

ILLUSTRATIVE EXAMPLE

A line source $Q^* = 52$ W/cm is located at (5, 2) in the element shown in Figure 9.4. Determine the amount of Q^* allocated to each node.

The values of the a, b, and c constants are

$$\begin{array}{lll} a_i = 28, & a_j = 6, & a_k = -21 \\ b_i = -4, & b_j = 1, & b_k = 3 \\ c_i = -1, & c_j = -3, & c_k = 4 \end{array}$$

The shape function equations can be written after recalling that

$$a_i + a_j + a_k = 2A = 13$$

The equations are

$$N_i = \tfrac{1}{13}[28 - 4x - y]$$
$$N_j = \tfrac{1}{13}[6 + x - 3y]$$
$$N_k = \tfrac{1}{13}[-21 + 3x + 4y]$$

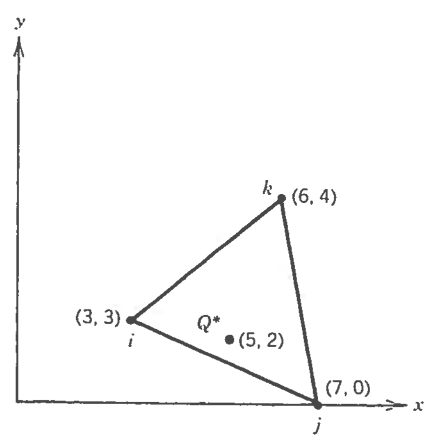

Figure 9.4. A point source in a triangular element.

Substituting $x = X_0 = 5$ and $y = Y_0 = 2$ produces

$$N_i = \tfrac{1}{13}[28 - 4(5) - 2] = \tfrac{6}{13}$$
$$N_j = \tfrac{1}{13}[6 + 5 - 3(2)] = \tfrac{5}{13}$$
$$N_k = \tfrac{1}{13}[-21 + 3(5) + 4(2)] = \tfrac{2}{13}$$

The value of Q^* is allotted to nodes i, j, and k by the fractions $\tfrac{6}{13}$, $\tfrac{5}{13}$ and $\tfrac{2}{13}$, respectively. Therefore,

$$\{f_Q^{(e)}\} = \frac{52}{13}\begin{Bmatrix} 6 \\ 5 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 20 \\ 8 \end{Bmatrix}$$

The best location for a source or sink is at a node. This location changes the result given in (9.38). If we assume that the source is at node j (Figure 9.5), then $N_i = N_k = 0$ and

$$\{f_Q^{(e)}\} = Q^* \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \tag{9.39}$$

The magnitude of Q^*, however, must be modified when the source (sink) is shared by more than one element. The magnitude of the source is divided among the elements joining at the node. The source is allocated according to the ratio of the angle in the element to 360. The correct equation for element (e) in Figure 9.5 is

$$\{f_Q^{(e)}\} = \frac{\alpha Q^*}{360} \begin{Bmatrix} C \\ 1 \\ 0 \end{Bmatrix} \tag{9.40}$$

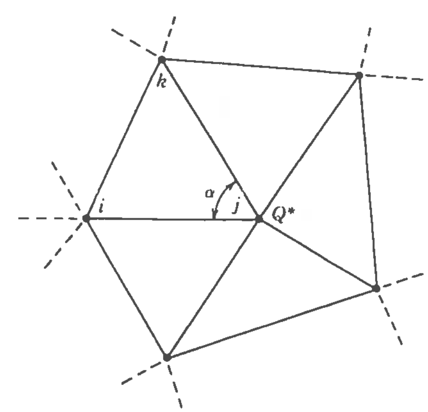

Figure 9.5. A point source at a node.

There is no need to evaluate α for the various elements. When the equations are assembled using the direct stiffness method, the element contributions at this node add to Q^*. An easier procedure for implementing a node source is to add the value of Q^* to the row of $\{F\}$ corresponding to the node number. A source is positive whereas the sink has a negative sign.

PROBLEMS

9.1 The boundary condition around the outside of a rectangular region follows. Determine the magnitude and sign for M and S on each of the sides; the sides are labeled as shown in Figure P9.1.

(a) $D_x \dfrac{\partial \phi}{\partial x} = 6\phi_b - 3,$ side 2

(b) $D_y \dfrac{\partial \phi}{\partial y} = -4\phi_b + 6,$ side 1

(c) $D_x \dfrac{\partial \phi}{\partial x} = 5\phi_b + 2,$ side 4

(d) $D_y \dfrac{\partial \phi}{\partial y} = 8\phi_b - 4,$ side 3

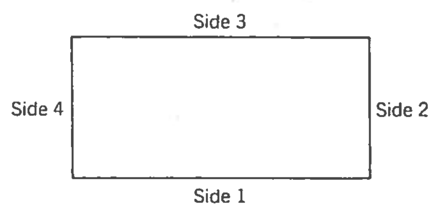

Figure P9.1

9.2 Evaluate the integral in (9.14) for:

(a) Side jk of a rectangular element.
(b) Side km of a rectangular element.
(c) Side im of a rectangular element.
(d) Side jk of a triangular element.
(e) Side ik of a triangular element.

9.3 Evaluate the integral in (9.13) for

(a) Side jk of a rectangular element.
(b) Side km of a rectangular element.
(c) Side im of a rectangular element.

(d) Side jk of a triangular element.
(e) Side ik of a triangular element.

9.4–9.8 Evaluate $\{f_Q^{(e)}\}$ for a point source, $Q^* = 40$ W/cm, located at point A for the corresponding triangular element in Problems 5.7–5.11.

9.9–9.13 Evaluate $\{f_Q^{(e)}\}$ for a point source, $Q^* = 40$ W/cm, located at point B for the corresponding rectangular element in Problems 5.12 to 5.16.