Problem 3.7

$$\int_0^H WQ\,dx = \int_{X_r}^{X_s} (N_s Q)^{(e)} dx + \int_{X_s}^{X_t} (N_s Q)^{(e+1)} dx$$

$$N_s^{(e)} = \frac{x - X_r}{L} \text{ , eq (3.14)}, \quad Q^{(e)} = \left(\frac{X_s - x}{L}\right) Q_r + \left(\frac{x - X_r}{L}\right) Q_s \quad \text{eq (3.13)}$$

$$\int_{X_r}^{X_s} \left(\frac{x - X_r}{L}\right)\left[\left(\frac{X_s - x}{L}\right) Q_r + \left(\frac{x - X_r}{L}\right) Q_s\right] dx$$

$$= \frac{Q_r}{L^2}\int_{X_r}^{X_s} (x X_s - X_r X_s + x X_r - x^2)\,dx + \frac{Q_s}{L^2}\int_{X_r}^{X_s}\left(x^2 - 2x X_r + X_r^2\right)dx$$

$$= \frac{Q_r}{L^2}\left(\frac{x^2 X_s}{2} - X_r X_s x + \frac{x^2 X_r}{2} - \frac{x^3}{3}\right\rvert_{X_r}^{X_s} + \frac{Q_s}{L^2}\left(\frac{x^3}{3} - \frac{2x^2 X_r}{2} + X_r^2 x\right\rvert_{X_r}^{X_s}$$

$$= \frac{Q_r}{L^2}\left(\frac{X_s^3}{2} - X_r X_s^2 + \frac{X_s^2 X_r}{2} - \frac{X_s^3}{3} - \frac{X_r^2 X_s}{2} + X_r^2 X_s - \frac{X_r^3}{2} + \frac{X_r^3}{3}\right)$$

$$+ \frac{Q_s}{L^2}\left(\frac{X_s^3}{3} - \frac{\cancel{2} X_s^2 X_r}{\cancel{2}} + X_r^2 X_s - \frac{X_r^3}{3} + X_r^3 - X_r^3\right)$$

$$= \frac{Q_r}{L^2}\left(\frac{X_s^3}{6} - \frac{X_s^2 X_r}{2} + \frac{X_r^2 X_s}{2} - \frac{X_r^3}{6}\right) + \frac{Q_s}{L^2}\left(\frac{X_s^3}{3} - X_s^2 X_r + X_r^2 X_s - \frac{X_r^3}{3}\right)$$

$$= \frac{Q_r}{6L^2}\left(X_s^3 - 3X_s^2 X_r + 3X_r^2 X_s - X_r^3\right) + \frac{Q_s}{3L^2}\left(X_s^3 - 3X_s^2 X_r + 3X_r^2 X_s - X_r^3\right)$$

$$= \frac{Q_r}{6L^2}(X_s - X_r)^3 + \frac{1}{3L^2}(X_s - X_r)^3 = \frac{Q_r L}{6} + \frac{Q_s L}{3}$$

Problem 3.7 (con't)

$$\int_{X_s}^{X_t} (N_s Q)^{(e+1)} dx \quad , \quad N_s^{(e+1)} = \frac{X_t - X}{L} , \quad Q^{(e+1)} = \left(\frac{X_t - X}{L}\right) Q_s + \left(\frac{X - X_s}{L}\right) Q_t$$

(3.20)

$$\int_{X_s}^{X_t} N_s Q \, dx = \int_{X_s}^{X_t} \left(\frac{X_t - X}{L}\right)\left[\left(\frac{X_t - X}{L}\right) Q_s + \left(\frac{X - X_s}{L}\right) Q_t\right] dx$$

$$= \frac{Q_s}{L^2} \int_{X_s}^{X_t} \left(\frac{X_t - X}{L}\right)^2 dx + \frac{Q_t}{L^2} \int_{X_s}^{X_t} \left(\frac{X_t - X}{L}\right)\left(\frac{X - X_s}{L}\right) dx$$

$$= \frac{Q_s L^3}{3L^2} + \frac{Q_t L^3}{6L^2} = \frac{Q_s L}{3} + \frac{Q_t L}{6}$$

using the same quantity for L^3 that occurred in the previous part.

$$\int W Q \, dx = \frac{Q_r L^{(e)}}{6} + Q_s\left(\frac{L^{(e)}}{3} + \frac{L^{(e+1)}}{3}\right) + \frac{Q_t L^{(e+1)}}{6}$$

when $L^{(e)} = L^{(e+1)}$

$$\int W Q \, dx = \frac{L}{6}\left(Q_r + 4 Q_s + Q_t\right)$$