

Point Estimation

- A sample was collected: $X_1, X_2, \dots, X_n$
- We suspect that sample was drawn from a random variable distribution $f(x)$
- $f(x)$ has k parameters that we do not know
- Point estimates are estimates of the parameters of the $f(x)$ describing the population based on the sample
 - For exponential PDF: $f(x) = \lambda \exp(-\lambda x)$ one wants to estimate λ
 - For Bernoulli PDF: $p^x(1-p)^{1-x}$ one wants to estimate p
 - For normal PDF one wants to estimate both μ and σ
- Point estimates are uncertain: therefore, we can talk of averages and standard deviations of point estimators

Method of Maximum Likelihood for point estimation

Maximum Likelihood Estimators

- Suppose that X is a random variable with probability distribution $f(x, \theta)$, where θ is a single unknown parameter. Let $x_1, x_2, \dots, x_n$ be the observed values in a random sample of size n . Then the **likelihood function** of the sample is the probability to get it in a random variable with PDF $f(x, \theta)$:

$$L(\theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdot \dots \cdot f(x_n, \theta) \quad (7-9)$$

- Note that the likelihood function is now a function of only the unknown parameter θ . The **maximum likelihood estimator** (MLE) of θ is the value of θ that maximizes the likelihood function $L(\theta)$.
- Usually, it is easier to work with **logarithms**: $l(\theta) = \ln L(\theta)$

Exponential MLE:

$$f(x_i) = \lambda e^{-\lambda x_i}$$

$$L(\lambda) = P(x_1, x_2, \dots, x_n | \lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i} =$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^n x_i}$$

$$\ln L(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^n x_i$$

$$\frac{d \ln L(\lambda)}{d \lambda} = \frac{n}{\lambda} - \sum x_i = 0$$

$$\hat{\lambda} = \frac{n}{\sum x_i} = \frac{1}{\bar{X}}$$

Same as
1st moment
estimator

Example 7-11: Exponential MLE

Let X be an exponential random variable with parameter λ . The likelihood function of a random sample of size n is:

$$L(\lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^n x_i}$$

$$\ln L(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^n x_i$$

$$\frac{d \ln L(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i = 0$$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{X}} \quad (\text{same as moment estimator})$$

Bernoulli: MLE

$$f(x, p) = p^x (1-p)^{1-x}$$

$$L(p) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i} =$$

$$= p^{\sum x_i} (1-p)^{n - \sum x_i}$$

$$\ln L(p) = (\sum x_i) \ln p + (n - \sum x_i) \ln(1-p)$$

$$\frac{d \ln L(p)}{dp} = \frac{\sum x_i}{p} - \frac{n - \sum x_i}{1-p} = 0 \quad \text{at } \hat{p}$$

$$0 = \frac{(1 - \cancel{\hat{p}}) \sum x_i - \hat{p} (n - \cancel{\sum x_i})}{\hat{p} (1 - \hat{p})} \quad \hat{p} = \frac{\sum_{i=1}^n x_i}{n}$$

Example 7-9: Bernoulli MLE

Let X be a Bernoulli random variable. The probability mass function is $f(x;p) = p^x(1-p)^{1-x}$, $x = 0, 1$ where P is the parameter to be estimated. The likelihood function of a random sample of size n is:

$$L(p) = p^{x_1}(1-p)^{1-x_1} \cdot p^{x_2}(1-p)^{1-x_2} \cdot \dots \cdot p^{x_n}(1-p)^{1-x_n}$$

$$= \prod_{i=1}^n p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i}$$

$$\ln L(p) = \left(\sum_{i=1}^n x_i \right) \ln p + \left(n - \sum_{i=1}^n x_i \right) \ln(1-p)$$

$$\frac{d \ln L(p)}{dp} = \frac{\sum_{i=1}^n x_i}{p} - \frac{(n - \sum_{i=1}^n x_i)}{(1-p)} = 0$$

$$\hat{p} = \frac{\sum_{i=1}^n x_i}{n} \text{ (same as moment estimator)}$$

Normal MLE for μ

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$L(\mu, \sigma) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(-\frac{\sum (x_i - \mu)^2}{2\sigma^2}\right)$$

$$\ln L(\mu, \sigma) = -n \ln(\sigma\sqrt{2\pi}) -$$
$$- \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$$

$$\frac{d \ln L(\mu, \sigma)}{d\mu} = \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) = 0 \text{ at } \hat{\mu}$$
$$\hat{\mu} = \frac{\sum_{i=1}^n x_i}{n}$$

Example 7-10: Normal MLE for μ

Let X be a normal random variable with unknown mean μ and variance σ^2 . The likelihood function of a random sample of size n is:

$$\begin{aligned} L(\mu) &= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{(2\sigma^2)}} \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{\frac{-1}{2\sigma^2} \sum_{i=1}^n (x_i-\mu)^2} \\ \ln L(\mu) &= \frac{-n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \\ \frac{d \ln L(\mu)}{d\mu} &= \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) = 0 \\ \hat{\mu} &= \frac{\sum_{i=1}^n x_i}{n} = \bar{X} \text{ (same as moment estimator)} \end{aligned}$$

Example 7-11: Normal MLE for σ^2

Let X be a normal random variable with the estimate of mean μ determined by MLE (see the previous slide) and an **unknown variance σ^2** . The likelihood function of a random sample of size n is:

$$\begin{aligned} L(\sigma) &= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2} \\ \ln L(\sigma) &= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \\ \frac{d \ln L(\sigma)}{d\sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (x_i - \mu)^2 = 0 \\ \widehat{\sigma^2} &= \frac{\sum_{i=1}^n (x_i - \mu)^2}{n} \quad (\text{biased estimator}) \end{aligned}$$

MLE for Poisson distribution

$$\begin{aligned} f(x_1, \dots, x_n | \lambda) &= \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \dots \frac{e^{-\lambda} \lambda^{x_n}}{x_n!} \\ &= \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{x_1! \dots x_n!} \end{aligned}$$

$$\log f(x_1, \dots, x_n | \lambda) = -n\lambda + \sum_{i=1}^n x_i \log \lambda - \log c$$

where $c = \prod_{i=1}^n x_i!$ does not depend on λ , and

$$\frac{d}{d\lambda} \log f(x_1, \dots, x_n | \lambda) = -n + \frac{\sum_{i=1}^n x_i}{\lambda}$$

By equating to zero, we obtain that the maximum likelihood estimate $\hat{\lambda}$ equals

$$\hat{\lambda} = \frac{\sum_{i=1}^n x_i}{n}$$

Credit: XKCD
comics

WHY ARE THERE SLAVES IN THE BIBLE

WHY DO TWINS HAVE DIFFERENT FINGERPRINTS
WHY ARE AMERICANS AFRAID OF DRAGONS

WHY IS HTTPS CROSSED OUT IN RED
WHY IS THERE A LINE THROUGH HTTPS
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT

WHY AREN'T MY
ARMS GROWING



WHY ARE THERE WEEKS
WHY DO I FEEL DIZZY

QUESTIONS

FOUND IN GOOGLE AUTOCOMPLETE

WHY AREN'T ECONOMISTS RICH

WHY DO AMERICANS CALL IT SOCCER

WHY ARE MY EARS RINGING

WHY ARE THERE SO MANY AVENGERS

WHY ARE THE AVENGERS FIGHTING THE X MEN

WHY IS WOLVERINE NOT IN THE AVENGERS

WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN

WHY IS PSYCHIC WEAK TO BUG

WHY DO CHILDREN GET CANCER

WHY IS POSEIDON ANGRY WITH ODYSSEUS

WHY IS THERE ICE IN SPACE

WHY ARE THERE ANTS IN MY LAPTOP

WHY IS EARTH TILTED

WHY IS SPACE BLACK

WHY IS OUTER SPACE SO COLD

WHY ARE THERE PYRAMIDS ON THE MOON

WHY IS NASA SHUTTING DOWN

WHY ARE THERE MALE AND FEMALE BIKES

WHY ARE THERE TINY SPIDERS IN MY HOUSE

WHY DO SPIDERS COME INSIDE

WHY ARE THERE HUGE SPIDERS IN MY HOUSE

WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE

WHY ARE THERE SPIDERS IN MY ROOM

WHY ARE THERE SO MANY SPIDERS IN MY ROOM

WHY DO SPIDER BITES ITCH

WHY IS DYING SO SCARY

WHY IS THERE NO GPS IN LAPTOPS

WHY DO KNEES CLICK

WHY AREN'T THERE E GRADES

WHY IS ISOLATION BAD

WHY DO BOYS LIKE ME

WHY DON'T BOYS LIKE ME

WHY IS THERE ALWAYS A JAVA UPDATE

WHY ARE THERE RED DOTS ON MY THIGHS

WHY IS LYING GOOD

WHY ARE THERE
GHOSTS



WHY IS THERE AN OWL IN MY BACKYARD

WHY IS THERE AN OWL OUTSIDE MY WINDOW

WHY IS THERE AN OWL ON THE DOLLAR BILL

WHY DO OWLS ATTACK PEOPLE

WHY ARE AK 47s SO EXPENSIVE

WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE

WHY ARE THERE GODS

WHY ARE THERE TWO SPOCKS

WHY IS MT VESUVIUS THERE

WHY DO THEY SAY T MINUS

WHY ARE THERE OBELISKS

WHY ARE WRESTLERS ALWAYS WET

WHY ARE OCEANS BECOMING MORE ACIDIC

WHY IS ARWEN DYING

WHY AREN'T MY QUAIL LAYING EGGS

WHY AREN'T MY QUAIL EGGS HATCHING

WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA

WHY ARE CIGARETTES LEGAL
WHY ARE THERE DUCKS IN MY POOL
WHY IS JESUS WHITE
WHY IS THERE LIQUID IN MY EAR
WHY DO Q TIPS FEEL GOOD
WHY DO GOOD PEOPLE DIE

WHY AREN'T
THERE GUNS IN
HARRY POTTER



WHY ARE ULTRASOUNDS IMPORTANT
WHY ARE ULTRASOUND MACHINES EXPENSIVE
WHY IS STEALING WRONG

WHY DO WHALES JUMP
WHY ARE WITCHES GREEN
WHY ARE THERE MIRRORS ABOVE BEDS

WHY DO I SAY UH
WHY IS SEA SALT BETTER
WHY ARE THERE TREES IN THE MIDDLE OF FIELDS

WHY IS THERE NOT A POKEMON MMO
WHY IS THERE LAUGHING IN TV SHOWS
WHY ARE THERE DOORS ON THE FREEWAY

WHY ARE THERE SO MANY SVCHOST.EXE RUNNING
WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA
WHY ARE THERE SCARY SOUNDS IN MINECRAFT

WHY IS THERE KICKING IN MY STOMACH
WHY ARE THERE TWO SLASHES AFTER HTTP
WHY ARE THERE CELEBRITIES

WHY DO SNAKES EXIST
WHY DO OYSTERS HAVE PEARLS
WHY ARE DUCKS CALLED DUCKS

WHY DO THEY CALL IT THE CLAP
WHY ARE KYLE AND CARTMAN FRIENDS
WHY IS THERE AN ARROW ON AANG'S HEAD

WHY ARE TEXT MESSAGES BLUE
WHY ARE THERE MUSTACHES ON CLOTHES
WHY ARE THERE MUSTACHES ON CARS

WHY ARE THERE MUSTACHES EVERYWHERE
WHY ARE THERE SO MANY BIRDS IN OHIO
WHY IS THERE SO MUCH RAIN IN OHIO

WHY IS OHIO WEATHER SO WEIRD
WHY ARE THERE MALE AND FEMALE BIKES
WHY ARE THERE BRIDESMAIDS

WHY DO DYING PEOPLE REACH UP
WHY AREN'T THERE VARICOSE ARTERIES
WHY ARE OLD KINGDOMS DIFFERENT

WHY ARE THERE SQUIRRELS
WHY IS PROGRAMMING SO HARD
WHY IS THERE A 0 OHM RESISTOR

WHY DO AMERICANS HATE SOCCER
WHY DO RHYMES SOUND GOOD
WHY DO TREES DIE

WHY IS THERE NO SOUND ON CNN

WHY AREN'T POKEMON REAL

WHY AREN'T BULLETS SHARP

WHY DO DREAMS SEEM SO REAL

WHY ARE THERE GPS IN LAPTOPS

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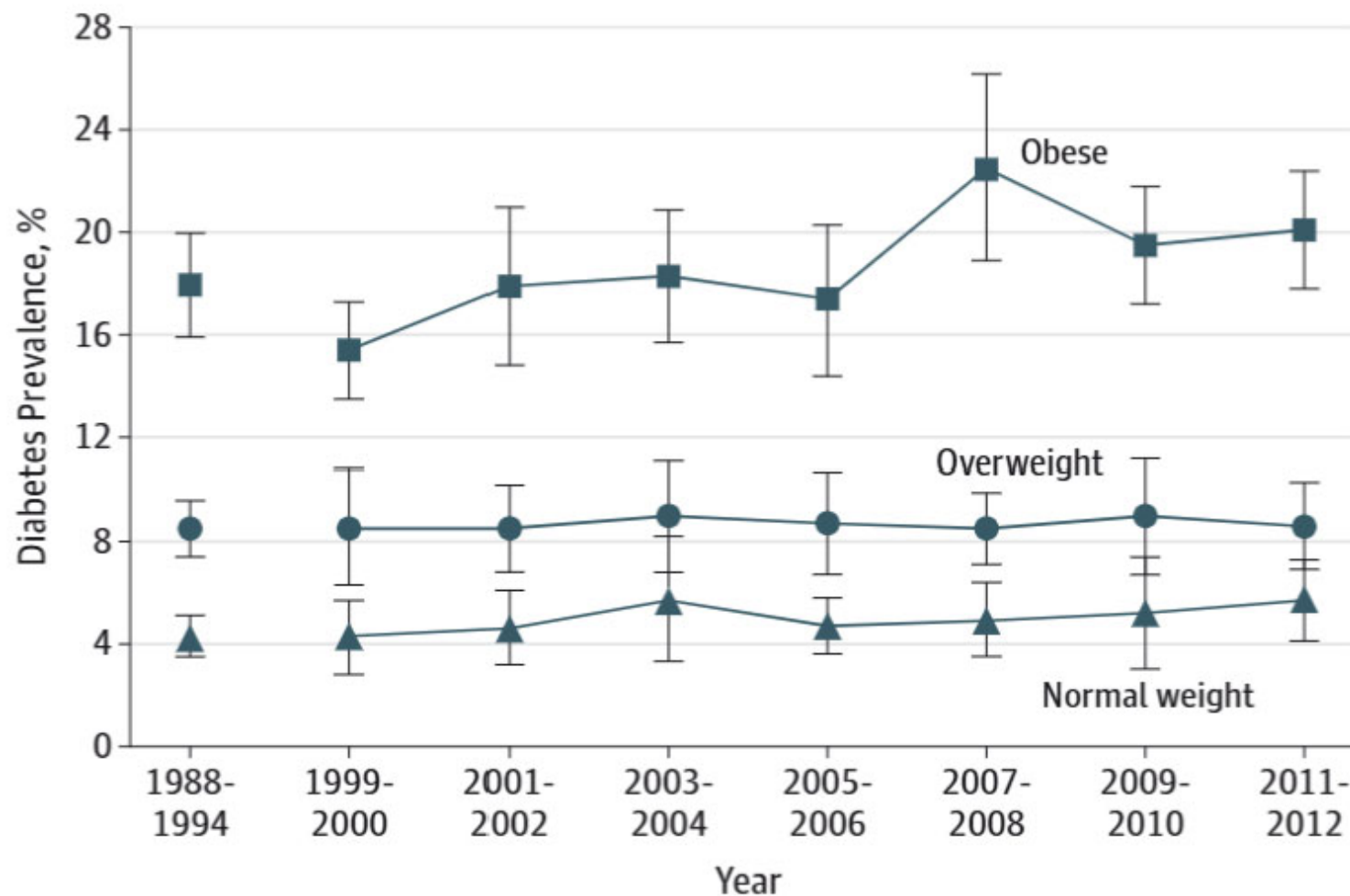
WHY IS THERE ALWAYS A JAVA UPDATE

WHY ARE THERE RED DOTS ON MY THIGHS

Confidence Intervals

- **Point estimates**: give the best single estimate of a parameter from sample data
- **Confidence intervals** (or interval estimate): give the upper and lower bounds within which a population parameter is with a given confidence level $1-\alpha$, say 95% ($\alpha=0.05$)

Figure 2. US Trends in Diabetes Prevalence per 100 Adults Aged 20 Years or Older by BMI Category



No. of participants

Obese	2324	727	732	815	820	1137	1302	1075
Overweight	2942	724	878	784	694	949	1009	852
Normal weight	3025	645	699	624	604	726	762	785

Menke A, Casagrande S, Geiss L, Cowie CC JAMA. 2015;314: 1021–1029

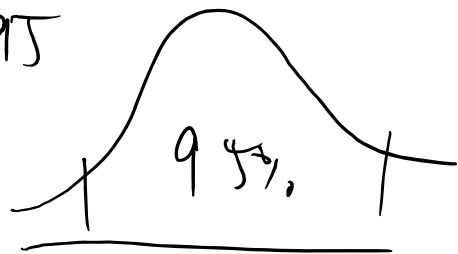
Two-sided confidence intervals

- Calculated based on the sample $X_1, X_2, \dots, X_n$
- Characterized by:
 - lower- and upper- confidence limits L and R
 - the confidence coefficient $1-\alpha$
- Objective: for two-sided confidence interval, find L and R such that
 - $\text{Prob}(\mu > R) = \alpha/2$
 - $\text{Prob}(\mu < L) = \alpha/2$
 - Therefore, $\text{Prob}(L < \mu < R) = 1-\alpha$
- For one-sided confidence interval, say, upper bound of μ , find R that
 - $\text{Prob}(\mu > R) = \alpha$
- **Assume standard deviation sigma is known**

Consider $1 - \alpha = 95\% = 0.95$

$$\alpha = 0.05; \quad \frac{\alpha}{2} = 0.025$$

$$z_{\alpha/2} = 1.96 \rightarrow \text{Prob}(Z > z_{\alpha/2}) = \frac{\alpha}{2}$$



$$\text{Prob}\left(-z_{\frac{\alpha}{2}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\text{Prob}\left(\bar{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

For one sided lower bound on μ

$$\text{Prob}\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \underline{\underline{z_{\alpha}}}\right) \rightarrow$$

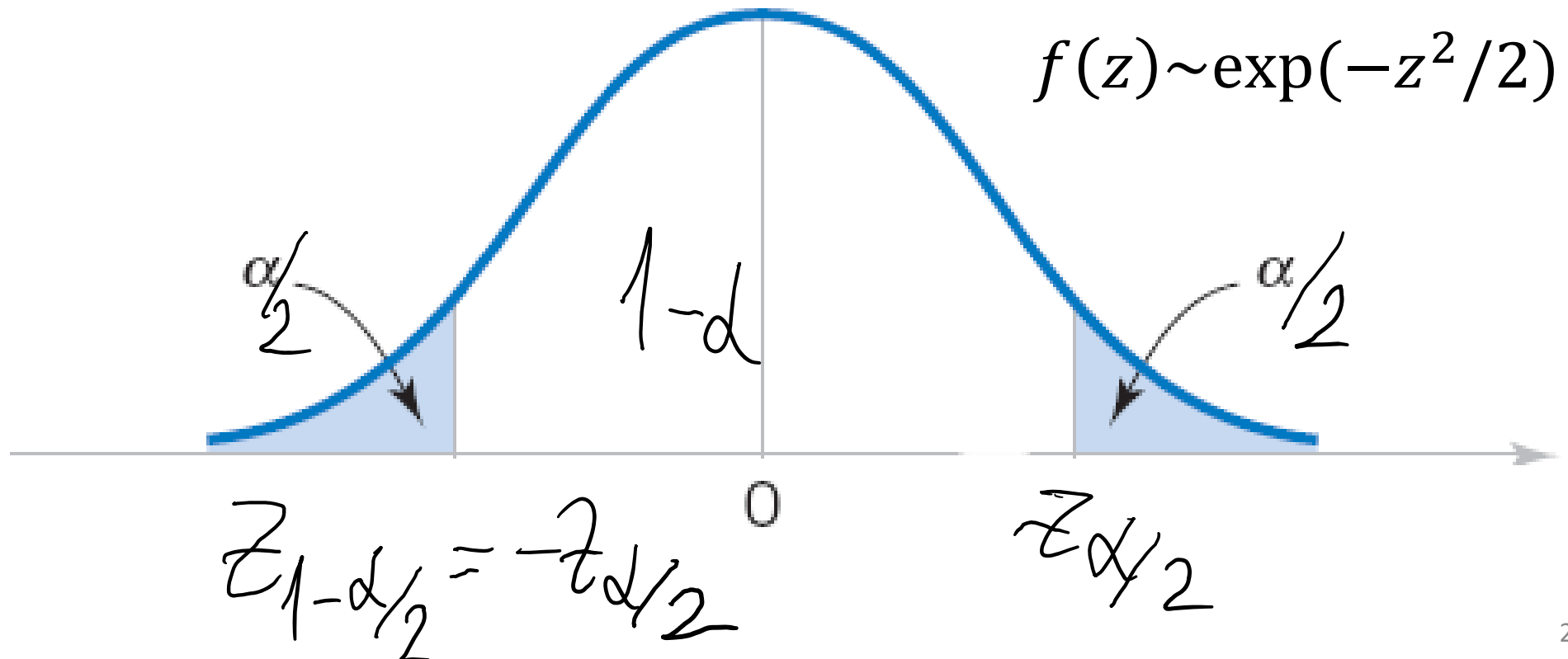
$$\mu > \bar{X} - z_{\alpha} \frac{\sigma}{\sqrt{n}}$$

$$z_{\alpha} = 1.65 <$$

$$z_{\alpha/2} = 1.96$$

Confidence Interval on the Population Mean, Variance Known

$$\bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$



Exercise

Ishikawa et al. (Journal of Bioscience and Bioengineering 2012) studied the force with which bacterial biofilms adhere to a solid surface.

Five measurements for a bacterial strain of *Acinetobacter* gave readings 2.69, 5.76, 2.67, 1.62, and 4.12 dyne-cm².

Assume that the standard deviation is known to be 0.66 dyne-cm²

- (a) Find 95% confidence interval for the mean adhesion force
- (b) If scientists want the width of the confidence interval to be below 0.55 dyne-cm² what number of samples should be?

Ishikawa et al. (Journal of Bioscience and Bioengineering 2012) studied the force with which bacterial biofilms adhere to a solid surface. Five measurements for a bacterial strain of Acinetobacter gave readings 2.69, 5.76, 2.67, 1.62, and 4.12 dyne-cm². Assume that the **standard deviation is known to be 0.66 dyne-cm²**

- (a) Find 95% confidence interval for the mean adhesion force
- (b) If scientists want the width of the confidence interval to be below 0.55 dyne-cm² what number of samples should be?

a) 95% CI for μ , $n = 5$ $\sigma = 0.66$ $\bar{x} = 3.372$, $z = 1.96$

$$\bar{x} - z\sigma / \sqrt{n} \leq \mu \leq \bar{x} + z\sigma / \sqrt{n}$$

$$3.372 - 1.96(0.66 / \sqrt{5}) \leq \mu \leq 3.372 + 1.96(0.66 / \sqrt{5})$$

$$2.79 \leq \mu \leq 3.95$$

b) Width is $2z\sigma / \sqrt{n} = 0.55$, therefore $n = [2z\sigma / 0.55]^2 = [2(1.96)(0.66) / 0.55]^2 = 22.13$
Round up to $n = 23$.

Matlab exercise

- 1000 labs measured average P53 gene expression using $n=20$ samples drawn from the Gaussian distribution with $\mu=3$; $\sigma=2$;
- Each lab found 95% confidence estimates of the population mean μ **based on its sample only**
- Count the number of labs, where the population mean lies **outside their bounds**
- You should get ~ 50 labs out of 1000 labs

How I did it

- `n=20; k_labs=1000;`
- `rand_table=2.*randn(n,k_labs)+3;`
- `sample_mean=mean(rand_table,1);`
- `CI_low=sample_mean-1.96.*2./sqrt(n);`
- `CI_high=sample_mean+1.96.*2./sqrt(n);`
- `k_above=sum(3>CI_high)`
- `k_below=sum(3<CI_low)`
- `figure; ndisp=100; errorbar(1:ndisp,
sample_mean(1:ndisp),
ones(ndisp,1).*1.96.*2./sqrt(n),'ko');`
- `hold on; plot(1:ndisp, 3.*ones(ndisp,1),'r-');`

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

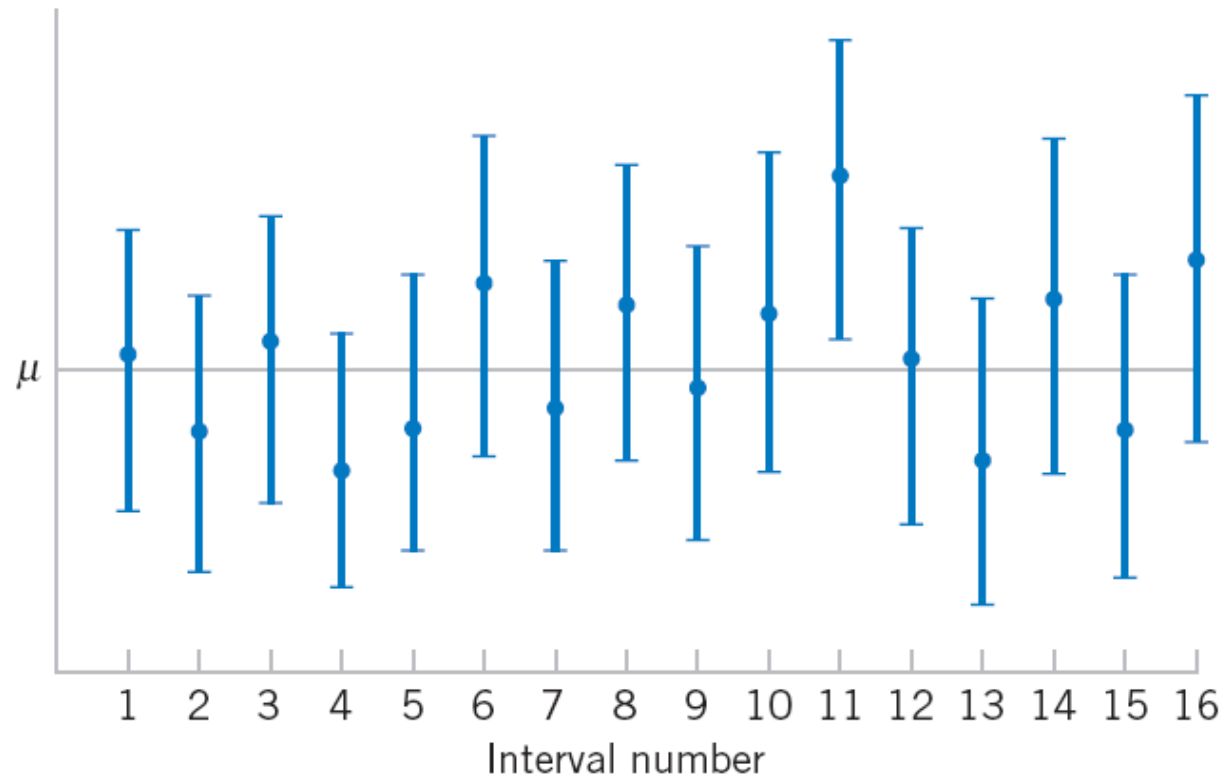


Figure 8-1 Repeated construction of a confidence interval for μ .

Figure 8-1 Repeated construction of a confidence interval for μ .

So far in estimating
confidence intervals for population mean μ
we assumed that the population variance σ^2
is known

Then (or when $n \gg 1$, say 20 and above)
one can use the Normal Distribution
to calculate confidence intervals

Q: What to do if the sample is small
& the population variance is **not known**?

A: Use the sample variance

$$s^2 = \frac{1}{n - 1} \sum (x_i - \bar{x})^2$$

but carefully:

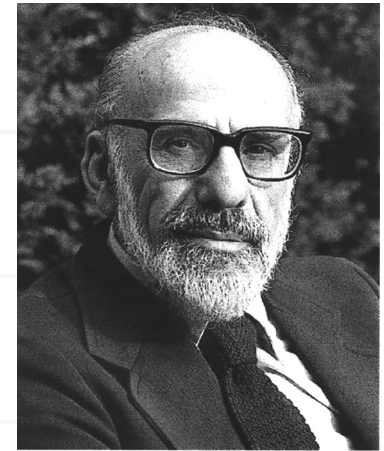
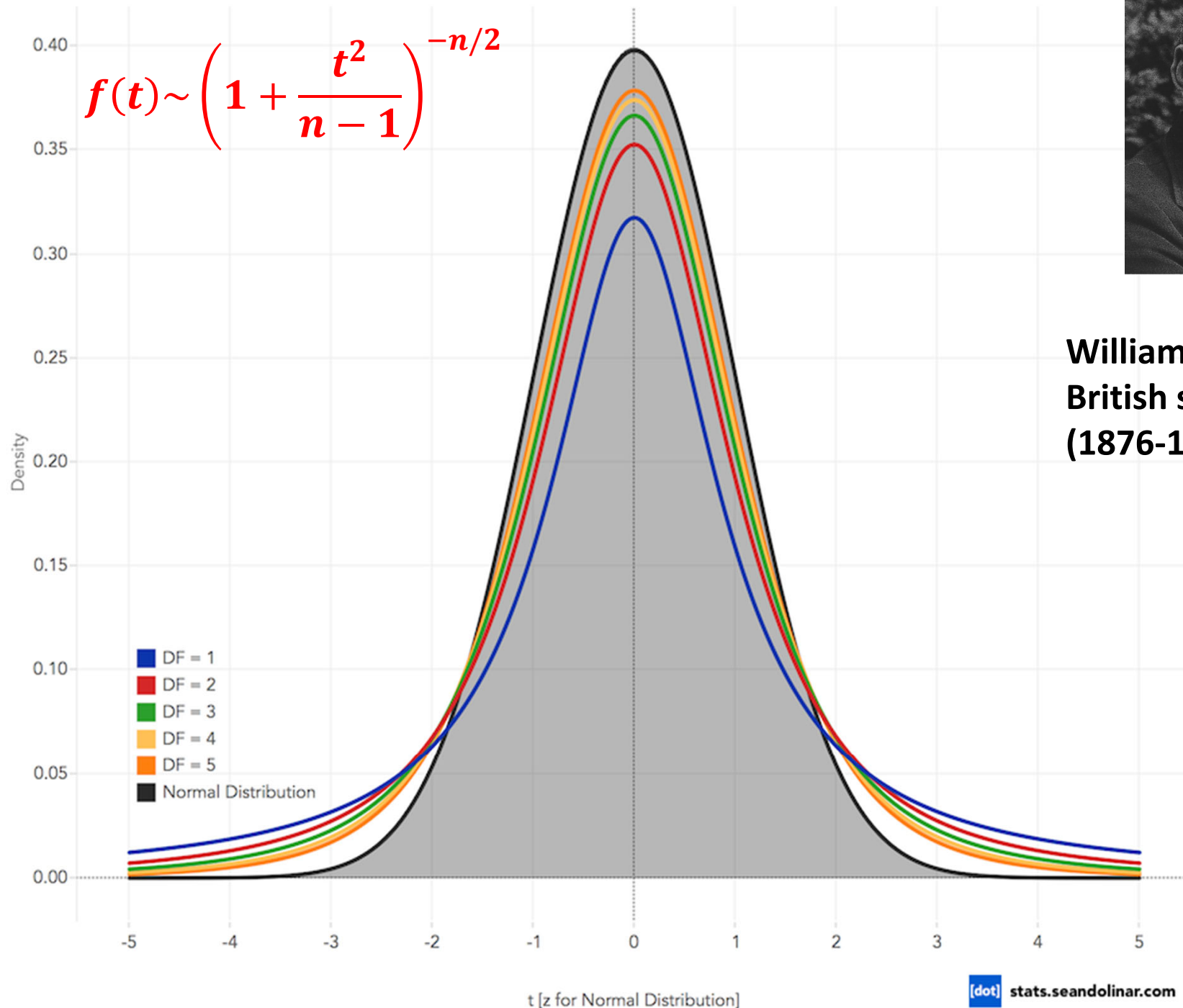
- Variable X has to be **normally distributed**
- **Student t-distribution** has to be used

instead of

the normal distribution (z-distribution).

Student's t-distribution

t-Distribution vs. Normal Distribution



William Sealy Gosset
British statistician
(1876-1937)

Play with Mathematica notebook

<http://demonstrations.wolfram.com/ComparingNormalAndStudentsTDistributions/>

By Gary McClelland

Another researcher at Guinness had previously published a paper containing trade secrets of the Guinness brewery. To prevent further disclosure of confidential information, Guinness prohibited its employees from publishing any papers regardless of the contained information. However, after pleading with the brewery and explaining that his mathematical and philosophical conclusions were of no possible practical use to competing brewers, he was allowed to publish them, but under a pseudonym ("Student"), to avoid difficulties with the rest of the staff. Thus, his most noteworthy achievement is now called Student's, rather than Gosset's, t-distribution.



William Sealy Gosset

(13 June 1876 – 16 October 1937)

was an English statistician, chemist and brewer who as Head Brewer of Guinness



Gosset had almost all his papers including “The probable error of a mean” (1908) published in Pearson's journal *Biometrika* under the pseudonym Student

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

$$\bar{x} - t_{\alpha/2, n-1} \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$$

Student's t distribution

$$f(t) \sim \left(1 + \frac{t^2}{n-1}\right)^{-n/2}$$

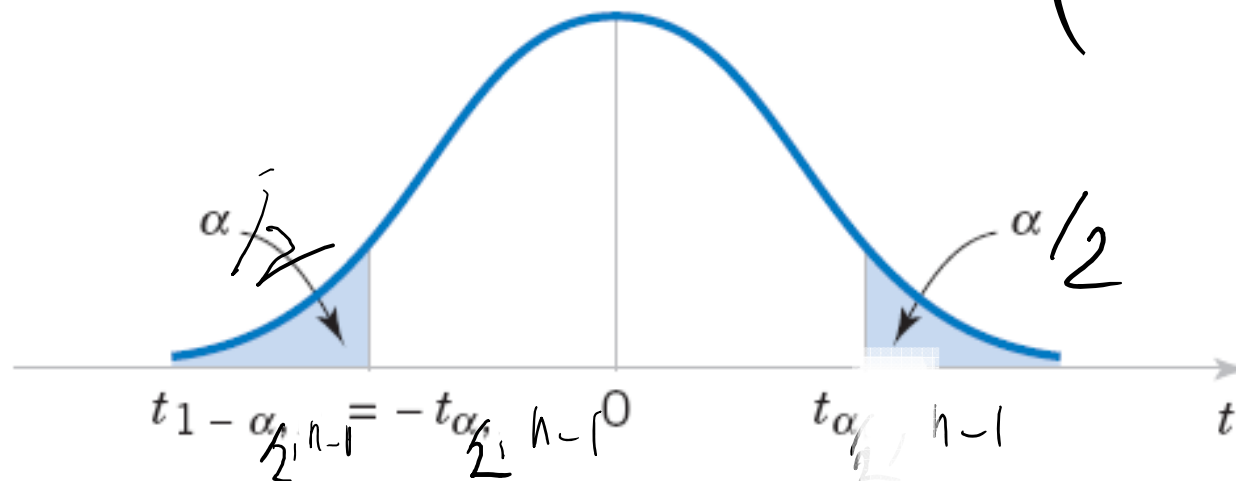


Figure 8-5 Percentage points of the t distribution.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

8-3.2 The t Confidence Interval on μ

(Eq. 8-16)

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2 , a **100(1 - α)% confidence interval on μ** is given by

$$\bar{x} - t_{\alpha/2, n-1}s/\sqrt{n} \leq \mu \leq \bar{x} + t_{\alpha/2, n-1}s/\sqrt{n} \quad (8-16)$$

where $t_{\alpha/2, n-1}$ is the upper 100 α /2 percentage point of the t distribution with $n - 1$ degrees of freedom.

One-sided confidence bounds on the mean are found by replacing $t_{\alpha/2, n-1}$ in Equation 8-16 with $t_{\alpha, n-1}$.