

Constant rate (Poisson) process

Memoryless property of the exponential distribution

$$P(X > t+s | X > s) = P(X > t)$$

$$\begin{aligned} P(X > t+s | X > s) &= \frac{P(X > t+s, X > s)}{P(X > s)} = \\ &= \frac{\exp(-r(t+s))}{\exp(-rs)} = \exp(-rt) = \\ &= P(X > t) \end{aligned}$$

Exponential is the only memoryless distribution

Survivorship curves and death rates

- Survivorship curve:

$$P(T > t) = \exp(-rt)$$

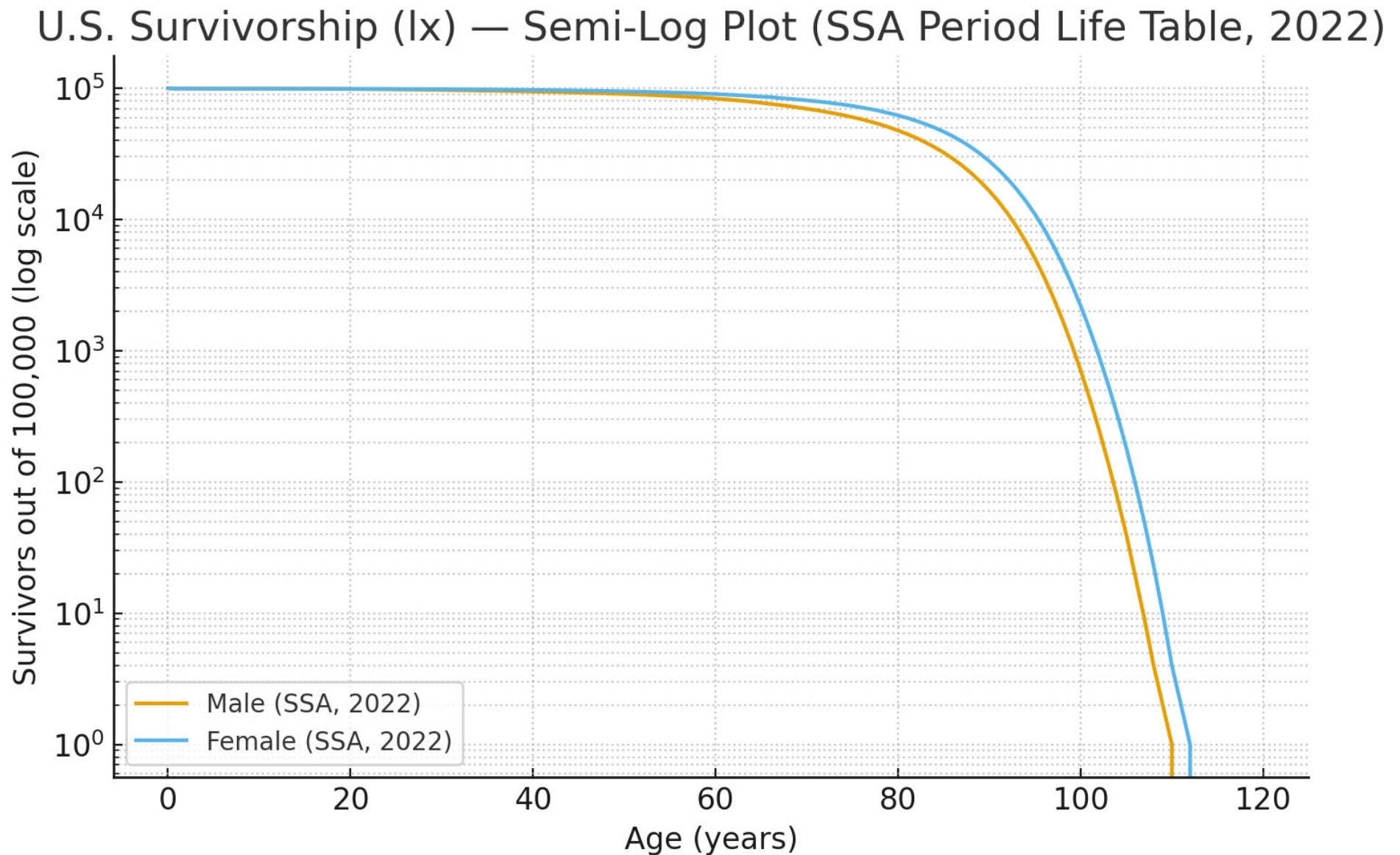
for a constant rate process

- Death rate conditioned on age:

$$P(t + \Delta t > T \mid T > t) = -\Delta t \cdot \frac{dP(T > t)/dt}{P(T > t)}$$
$$= r\Delta t$$

for a constant rate process

Survivorship curve: surviving fraction vs age

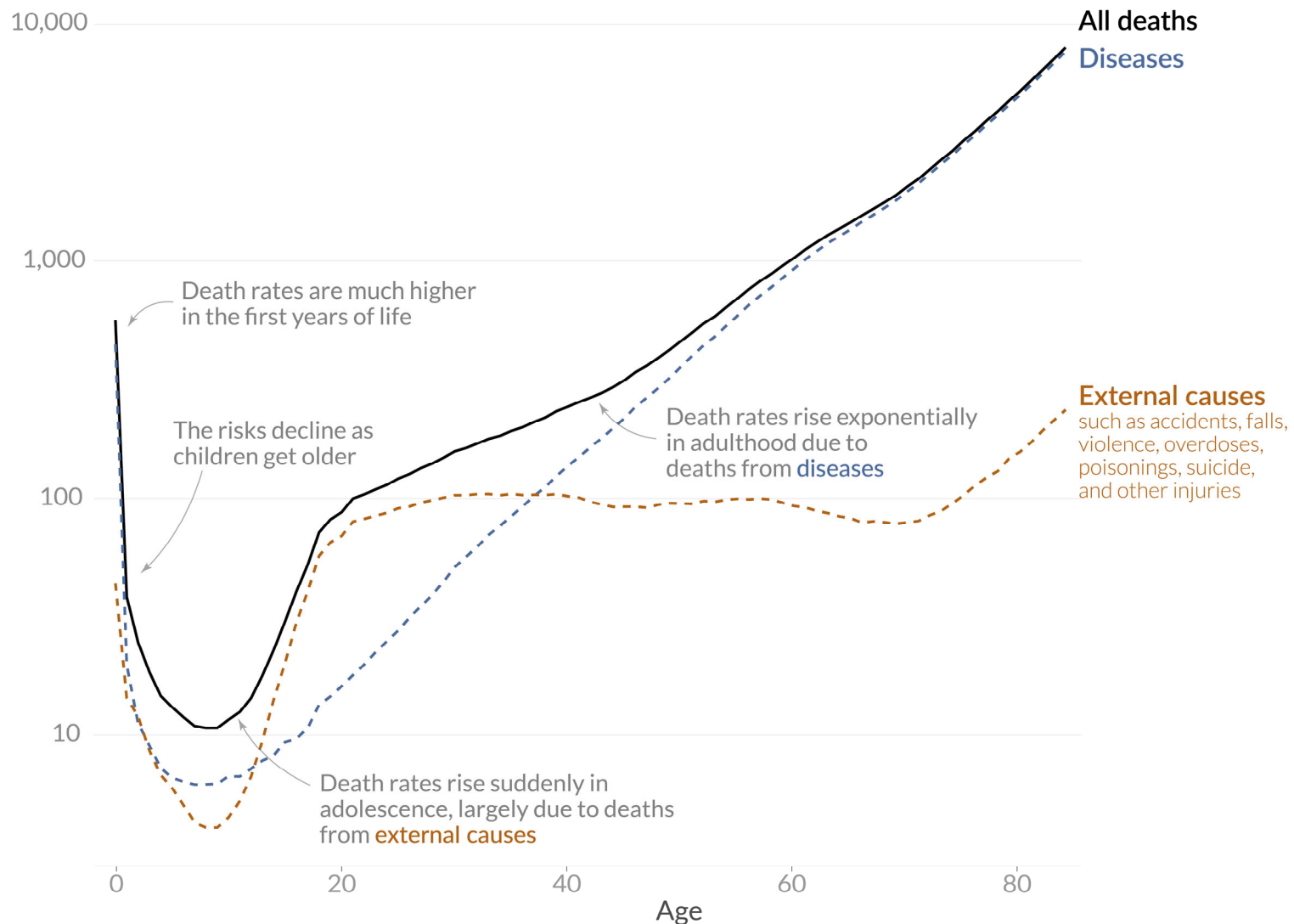


Death rates across ages

National data from the United States between 2018 and 2021.

Our World
in Data

Annual death rate, per 100,000 people (log scale)



Note: Period death rates using ICD-10 categories. 'Diseases' includes all categories except 'external causes' and 'signs, symptoms and abnormal findings'.

Source: United States Centers for Disease Control and Prevention, via CDC Wonder database

OurWorldinData.org — Research and data to make progress against the world's largest problems.

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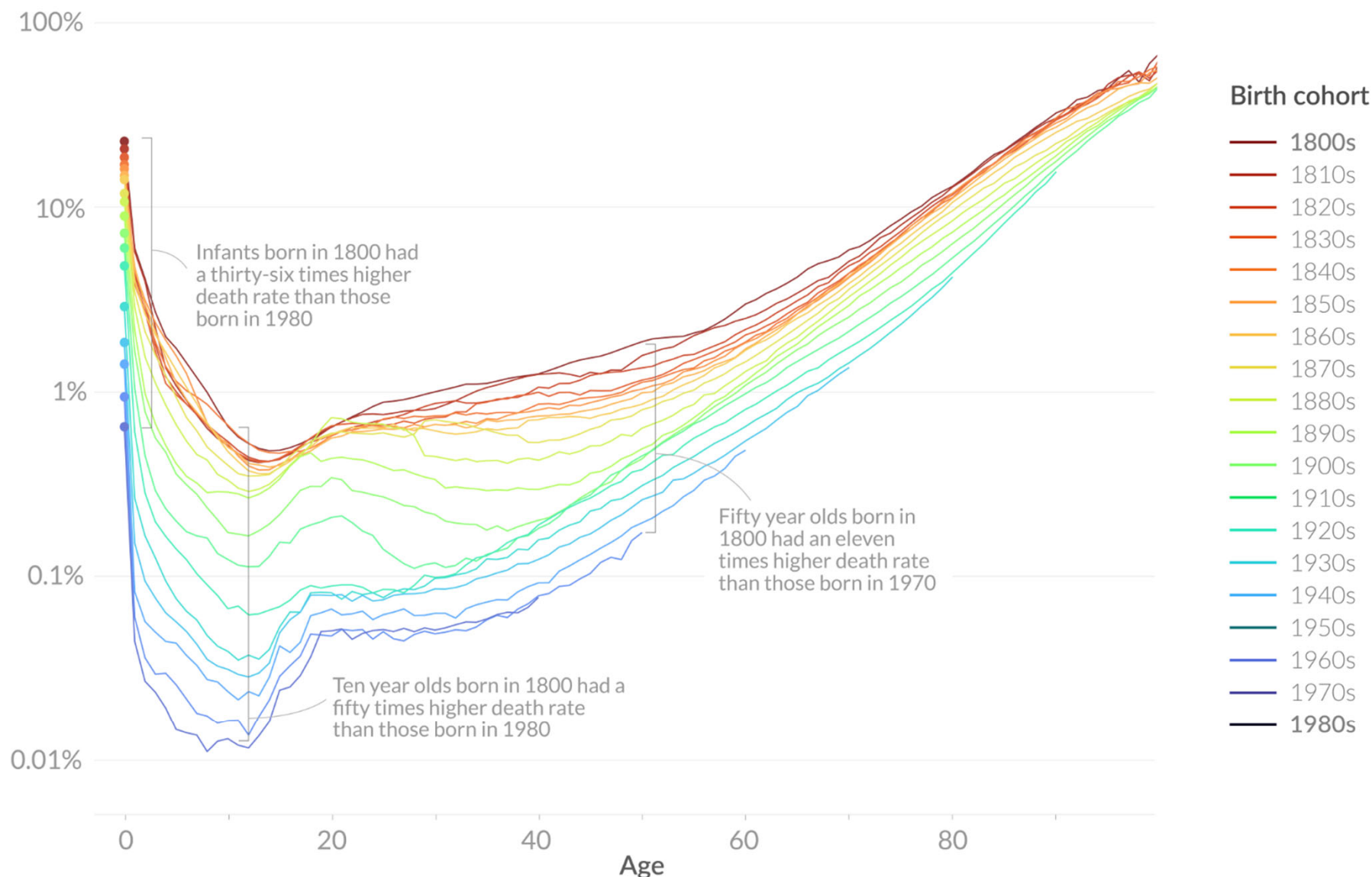
<https://ourworldindata.org/how-do-the-risks-of-death-change-as-people-age>

Death rates have declined across the lifespan

Cohort data from Sweden where long-term data is available. Annual death rates at age 0 are shown as dots.

Our World
in Data

Death rate (log scale)



Note: Lines begin for age groups who were included in the dataset, once data collection began. Lines end for those who have not yet reached a given age. Death rates above age 95 are not shown due to uncertainties.

Source: Human Mortality Database. Max Planck Institute for Demographic Research (Germany), University of California, Berkeley (USA), and French Institute for Demographic Studies (France).

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It is different for species/genera

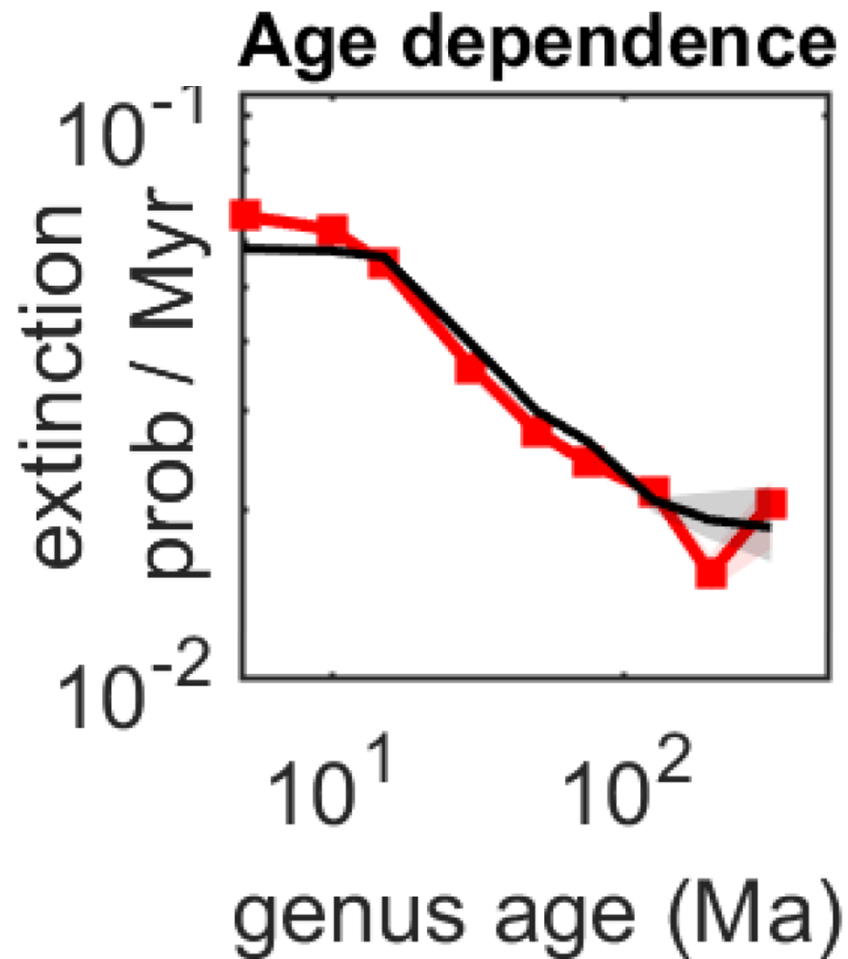
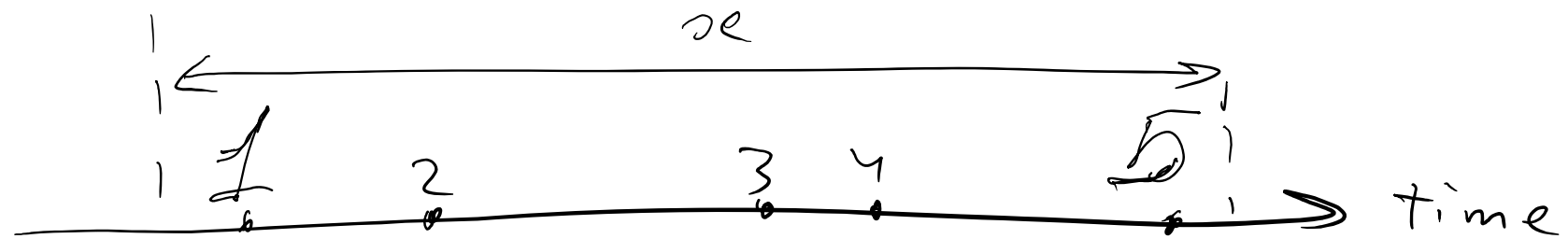


Figure from “**Survivorship Bias Explains Age-Dependent Extinction in Fossil Genera**”
Mikhail Tikhonov, Kim Sneppen, Stefan Bornholdt, and **Sergei Maslov**
<https://www.biorxiv.org/content/10.64898/2026.07.21.739780>

Erlang Distribution

- The Erlang distribution is a generalization of the exponential distribution.
- The exponential distribution models the time interval to the 1st event, while the
- Erlang distribution models the time interval to the k^{th} event, i.e., a sum of k exponentially distributed variables.
- The exponential, as well as Erlang distributions, is based on the constant rate (or Poisson) process.

Constant rate (Poisson) process



Events happen independently
from each other at

constant rate = r : $E[N_x] = rx$

- X follows Erlang distribution

$$P(X > x) = \sum_{n=0}^{r-1} P(N_x = n) =$$

$$= \sum_{n=0}^{r-1} \frac{(rx)^n}{n!} e^{-rx}$$

Erlang Distribution

Generalizes the Exponential Distribution:
waiting time between event 0 and **event k**
(constant rate process with rate=**r**)

$$P(X > x) = \sum_{m=0}^{k-1} \frac{e^{-rx} (rx)^m}{m!} = 1 - F(x)$$

Differentiating $F(x)$ we find that all terms in the sum except the last one cancel each other:

$$f(x) = \frac{r^k x^{k-1} e^{-rx}}{(k-1)!} \quad \text{for } x > 0 \quad \text{and } k = 1, 2, 3, \dots$$

Gamma Distribution

The random variable X with a probability density function:

$$f(x) = \frac{r^k x^{k-1} e^{-rx}}{\Gamma(k)}, \text{ for } x > 0 \quad (4-18)$$

has a gamma random distribution with parameters $r > 0$ and $k > 0$. If k is a positive integer, then X has an Erlang distribution.



$$f(x) = \frac{r^k x^{k-1} e^{-rx}}{\Gamma(k)}, \text{ for } x > 0$$

$$\int_0^{+\infty} f(x) dx = 1, \quad \text{Hence}$$

$$\Gamma(k) = \int_0^{+\infty} r^k x^{k-1} e^{-rx} dx = \int_0^{+\infty} y^{k-1} e^{-y} dy$$

Comparing with Erlang distribution
for integer k one gets

$$\Gamma(k) = (k-1)!$$

Gamma Function

The gamma function is the generalization of the factorial function for $r > 0$, not just non-negative integers.

$$\Gamma(k) = \int_0^{\infty} y^{k-1} e^{-y} dy, \quad \text{for } r > 0 \quad (4-17)$$

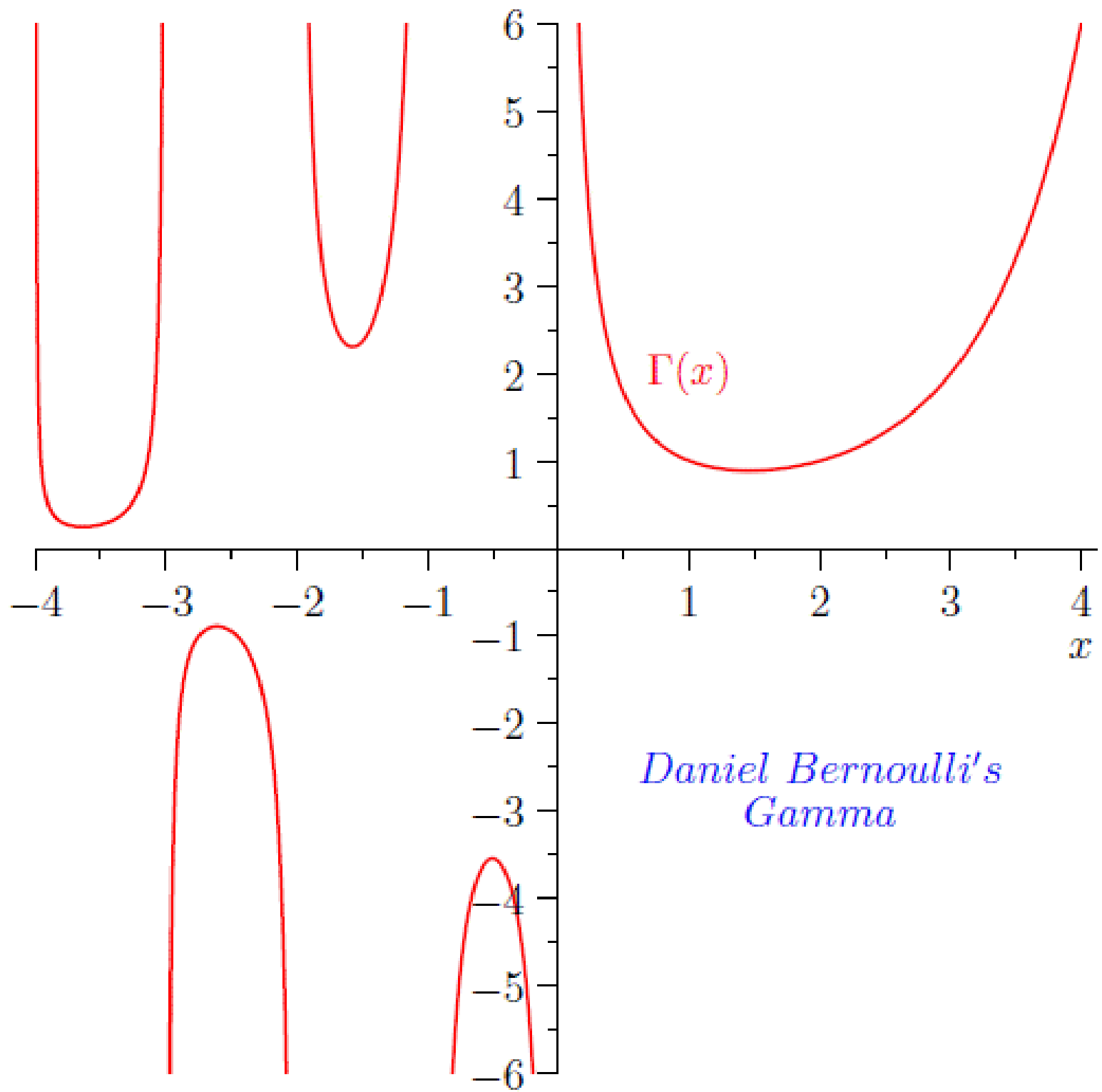
Properties of the gamma function

$$\Gamma(1) = 1$$

$$\Gamma(k) = (k-1)\Gamma(k-1) \quad \text{recursive property}$$

$$\Gamma(k) = (k-1)! \quad \text{factorial function}$$

$$\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}} = 1.77 = \left(-\frac{1}{2}\right)! \quad \text{interesting fact}$$



SOLO

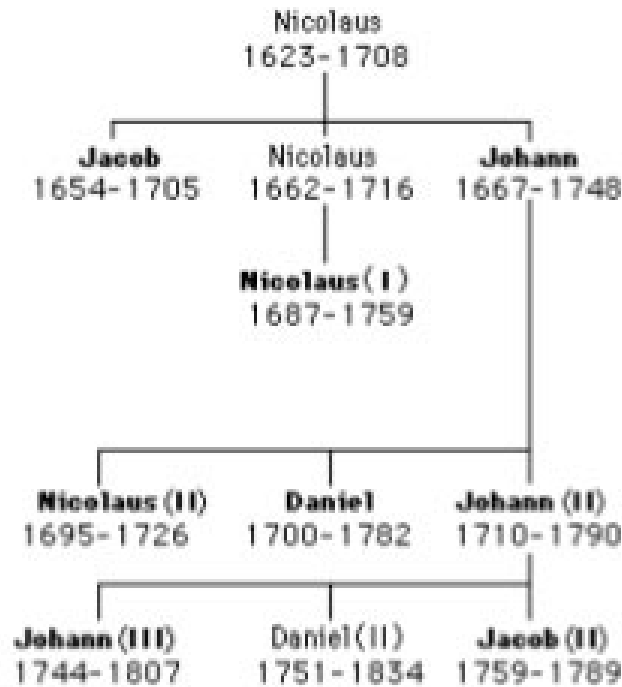
BERNOULLI FAMILY

Bernoulli trials

SOLO HERMELIN

<http://www.solohermelin.com>

The Bernoulli family



Those shown in **bold** above are in our archive

Read This



Jacob
1654-1705



Johann
1667-1748



Nicolaus II
1695-1720



Daniel
1700-1782



Johann II
1710-1790



Johann III
1744-1807



Jacob II
1759-1789

Gamma function

Mean & Variance of the Erlang and Gamma

- If X is an **Erlang** (or more generally **Gamma**) distributed **random variable** with parameters r and k ,

$$\mu = E(X) = k/r \quad \text{and} \quad \sigma^2 = V(X) = k/r^2 \quad (4-19)$$

- Generalization of exponential results:
 $\mu = E(X) = 1/r \quad \text{and} \quad \sigma^2 = V(X) = 1/r^2 \quad \text{or}$

Negative binomial results:

$$\mu = E(X) = k/p \quad \text{and} \quad \sigma^2 = V(X) = k(1-p) / p^2$$

Matlab exercise:

- Generate a sample of 100,000 variables with exponential distribution with $r = 0.1$ using `random('Exponential'...)` command
- Generate a sample of 100,000 variables with “Harry Potter” Gamma distribution with $r = 0.1$ and $k = 9 \frac{3}{4}$ (9.75)
- Calculate mean and compare it to $1/r$ (Exponential) and k/r (Gamma)
- Calculate standard deviation and compare it to $1/r$ (Exponential) and $\sqrt{k/r}$ (Gamma)
- Plot semilog-y plots of PDFs and CCDFs.

Matlab exercise: Exponential

- `Stats=100000; r=0.1;`
- `r2=random('Exponential', 1./r, Stats,1);`
%Matlab uses mean=1/r as a parameter
- `disp([mean(r2),1./r]); disp([std(r2),1./r]);`
- `step=0.1; [a,b]=hist(r2,0:step:max(r2));`
- `pdf_e=a./sum(a)./step;`
- `figure; subplot(1,2,1); semilogy(b,pdf_e,'ko-');`
- `X=0:0.01:100;`
- `for m=1:length(X);`
 `cdf_e(m)=sum(r2>X(m))./Stats;`
`end;`
- `subplot(1,2,2); semilogy(X,cdf_e,'ko-');`

Matlab exercise: Gamma

- `Stats=100000; r=0.1; k=9.75;`
- `r2=random('Gamma', k,1./r, Stats,1);`
- `disp([mean(r2),k./r]);`
- `disp([std(r2),sqrt(k)./r]);`
- `step=0.1; [a,b]=hist(r2,0:step:max(r2));`
- `pdf_g=a./sum(a)./step;`
- `figure;`
- `subplot(1,2,1); semilogy(b,pdf_g,'ko-'); hold on;`
- `x=0:0.01:max(r2); clear cdf_g;`
- `for m=1:length(x);`
- `cdf_g(m)=sum(r2>x(m))./Stats;`
- `end;`
- `subplot(1,2,2); semilogy(x,cdf_g,'rd-');`

Credit: XKCD
comics

WHY ARE THERE SLAVES IN THE BIBLE

WHY DO TWINS HAVE DIFFERENT FINGERPRINTS
WHY ARE AMERICANS AFRAID OF DRAGONS

WHY IS HTTPS CROSSED OUT IN RED
WHY IS THERE A LINE THROUGH HTTPS
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT

WHY AREN'T MY
ARMS GROWING



WHY ARE THERE WEEKS
WHY DO I FEEL DIZZY

QUESTIONS

FOUND IN GOOGLE AUTOCOMPLETE

WHY AREN'T ECONOMISTS RICH

WHY DO AMERICANS CALL IT SOCCER

WHY ARE MY EARS RINGING

WHY ARE THERE SO MANY AVENGERS

WHY ARE THE AVENGERS FIGHTING THE X MEN

WHY IS WOLVERINE NOT IN THE AVENGERS

WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN

WHY IS PSYCHIC WEAK TO BUG

WHY DO CHILDREN GET CANCER

WHY IS POSEIDON ANGRY WITH ODYSSEUS

WHY IS THERE ICE IN SPACE

WHY ARE THERE SWARMS OF GNATS

WHY IS THERE PHLEGM

WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN

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WHY DO CHILDREN GET CANCER

WHY IS POSEIDON ANGRY WITH ODYSSEUS

WHY ARE THERE ANTS IN MY LAPTOP

WHY ARE THERE
GHOSTS



WHY IS THERE AN OWL IN MY BACKYARD

WHY IS THERE AN OWL OUTSIDE MY WINDOW

WHY IS THERE AN OWL ON THE DOLLAR BILL

WHY DO OWLS ATTACK PEOPLE

WHY ARE AK 47s SO EXPENSIVE

WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE

WHY ARE THERE GODS

WHY ARE THERE TWO SPOCKS

WHY ARE CIGARETTES LEGAL

WHY ARE THERE DUCKS IN MY POOL

WHY IS JESUS WHITE

WHY IS THERE LIQUID IN MY EAR

WHY DO Q TIPS FEEL GOOD

WHY DO GOOD PEOPLE DIE

WHY AREN'T
THERE GUNS IN
HARRY POTTER



WHY ARE ULTRASOUNDS IMPORTANT

WHY ARE ULTRASOUND MACHINES EXPENSIVE

WHY IS STEALING WRONG

WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA

WHY IS EARTH TILTED

WHY IS SPACE BLACK

WHY IS OUTER SPACE SO COLD

WHY ARE THERE PYRAMIDS ON THE MOON

WHY IS NASA SHUTTING DOWN

WHY ARE THERE TINY SPIDERS IN MY HOUSE

WHY DO SPIDERS COME INSIDE

WHY ARE THERE HUGE SPIDERS IN MY HOUSE

WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE

WHY ARE THERE SPIDERS IN MY ROOM

WHY ARE THERE SO MANY SPIDERS IN MY ROOM

WHY DO SPIDER BITES ITCH

WHY IS DYING SO SCARY

WHY IS THERE NO GPS IN LAPTOPS

WHY DO KNEES CLICK

WHY AREN'T THERE E GRADES

WHY IS ISOLATION BAD

WHY IS SEX
SO IMPORTANT



WHY ARE THERE TINY SPIDERS IN MY HOUSE

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Continuous Probability Distributions

Normal or Gaussian Distribution



Normal or Gaussian Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$-\infty < x < \infty$$

is a **normal random variable**

with mean μ ,

and standard deviation σ

sometimes denoted as

$$N(\mu, \sigma)$$



Carl Friedrich Gauss (1777 –1855)
German mathematician

Normal Distribution

- The location and spread of the normal are independently determined by mean (μ) and standard deviation (σ)

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

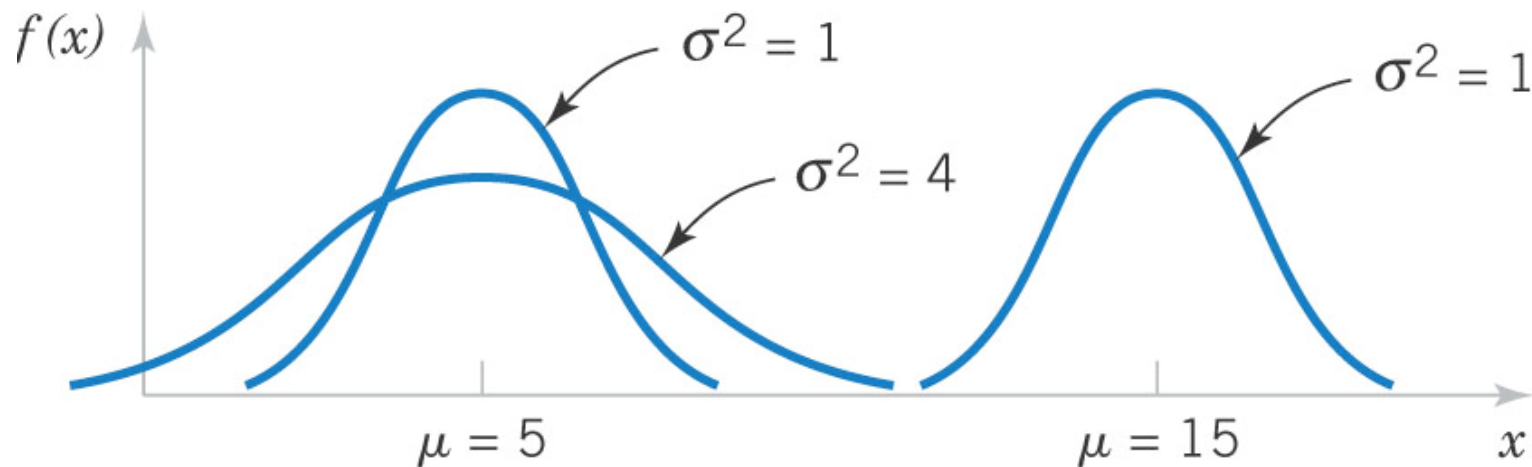


Figure 4-10 Normal probability density functions

Matlab exercise:
plot PDF of the Gaussian distribution
with **mu=3; sigma=2**
calculate mean, standard deviation
and variance,
Linear-y and Semilog-y plots of PDF

Hint:

Generate Standard normal
distribution using
randn(Stats,1) then
multiply and add using sigma, mu

Matlab exercise solution

- **Stats=100000;**
- **mu=3; sigma=2;**
- **r1=sigma.*randn(Stats,1)+mu;**
- **step=0.1;**
- **[a,b]=hist(r1,(mu-10.*sigma):step:(mu+10.*sigma));**
- **pdf_n=a./sum(a)./step;**
- **figure; subplot(1,2,1); plot(b,pdf_n,'ko-');**
- **subplot(1,2,2); semilogy(b,pdf_n,'ko-');**

Gaussian (Normal) distribution is very important because any sum of many independent random variables can be approximated with a Gaussian

Standard Normal Distribution

- A normal (Gaussian) random variable with

$$\mu = 0 \text{ and } \sigma^2 = 1$$

is called a **standard normal random variable** and is denoted as **Z**.

- The cumulative distribution function of a **standard normal random variable** is denoted as:

$$\Phi(z) = P(Z \leq z)$$

- Values are found in **Appendix A Table III** to **Montgomery and Runger textbook**

Standardizing

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma} \quad (4-10)$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a standard normal random variable.

Suppose X is a normal random variable with mean μ and variance σ^2 .

$$\text{Then, } P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P(Z \leq z) \quad (4-11)$$

where Z is a **standard normal random variable**, and

$z = \frac{(x - \mu)}{\sigma}$ is the z-value obtained by **standardizing** x .

The probability is obtained by using Appendix Table III

$$P(X < \mu - \sigma) = P(X > \mu + \sigma) = (1 - 0.68)/2 = 0.16 = 16\%$$

$$P(X < \mu - 2\sigma) = P(X > \mu + 2\sigma) = (1 - 0.95)/2 = 0.023 = 2.3\%$$

$$P(X < \mu - 3\sigma) = P(X > \mu + 3\sigma) = (1 - 0.997)/2 = 0.0013 = 0.13\%$$

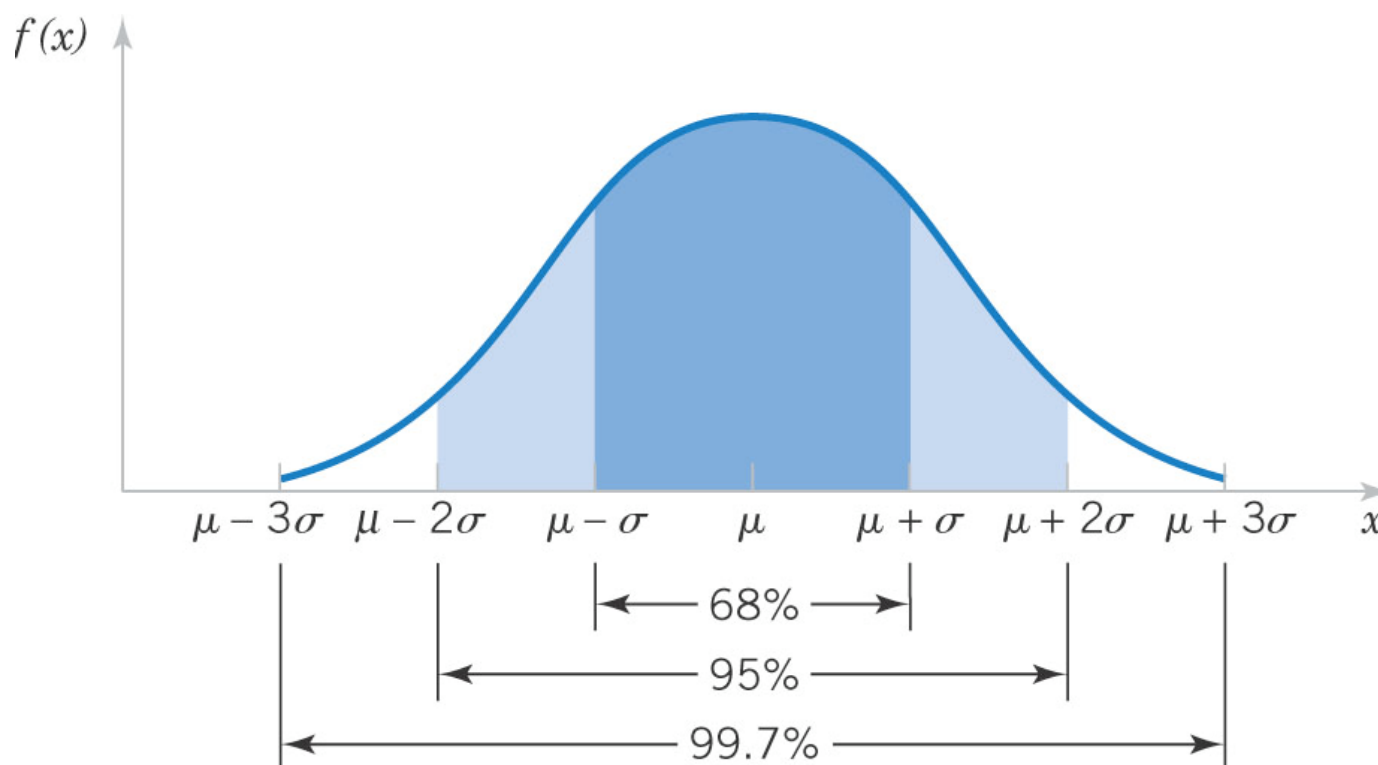


Figure 4-12 Probabilities associated with a normal distribution – well worth remembering to quickly estimate probabilities.

Range	The expected fraction of population inside the range	Approximate expected frequency outside the range	The approximate frequency for daily event
$\mu \pm 0.5\sigma$	0.382924922548026		2 in 3 Four or five times a week
$\mu \pm 1\sigma$	0.682689492137086		1 in 3 Twice a week
$\mu \pm 1.5\sigma$	0.866385597462284		1 in 7 Weekly
$\mu \pm 2\sigma$	0.954499736103642		1 in 22 Every three weeks
$\mu \pm 2.5\sigma$	0.987580669348448		1 in 81 Quarterly
$\mu \pm 3\sigma$	0.997300203936740		1 in 370 Yearly
$\mu \pm 3.5\sigma$	0.999534741841929		1 in 2149 Every six years
$\mu \pm 4\sigma$	0.999936657516334		1 in 15787 Every 43 years (twice in a lifetime)
$\mu \pm 4.5\sigma$	0.999993204653751		1 in 147160 Every 403 years (once in the modern era)
$\mu \pm 5\sigma$	0.999999426696856		1 in 1744278 Every 4776 years (once in recorded history)
$\mu \pm 5.5\sigma$	0.999999962020875		1 in 26330254 Every 72090 years (thrice in history of modern humankind)
$\mu \pm 6\sigma$	0.999999998026825		1 in 506797346 Every 1.38 million years (twice in history of humankind)
$\mu \pm 6.5\sigma$	0.999999999919680		1 in 12450197393 Every 34 million years (twice since the extinction of dinosaurs)
$\mu \pm 7\sigma$	0.999999999997440		1 in 390682215445 Every 1.07 billion years (four times in history of Earth)

Source: Wikipedia

DATA SCIENCE
DISCOVERY

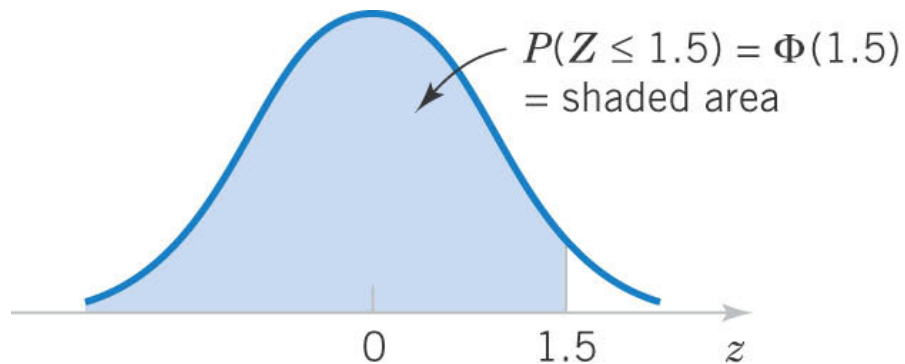
Human Impact of Probabilities
STAT 107: Data Science Discovery

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.523922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555670	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

Standard Normal Distribution Tables

Assume Z is a standard normal random variable.

Find $P(Z \leq 1.50)$. Answer: 0.93319



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
$\vdots$		$\vdots$		
1.5	0.93319	0.93448	0.93574	0.93699

Figure 4-13 Standard normal PDF

Find $P(Z \leq 1.53)$.

Answer: 0.93699

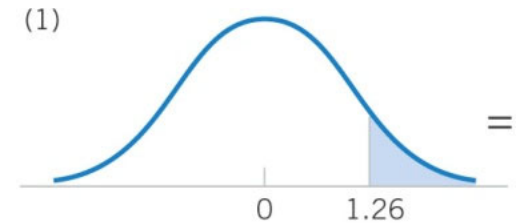
Find $P(Z \leq 0.02)$.

Answer: 0.50398

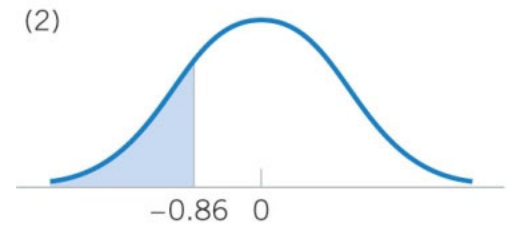
Table III from,
Appendix A in
Montgomery
& Runger

Standard Normal Exercises

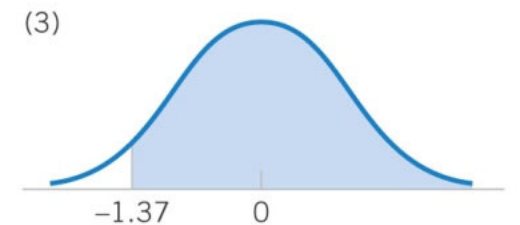
$$1. \quad P(Z > 1.26) = 1 - P(Z < 1.26) = 1 - \textcolor{red}{0.8962} = \textcolor{blue}{\underline{0.1038}}$$



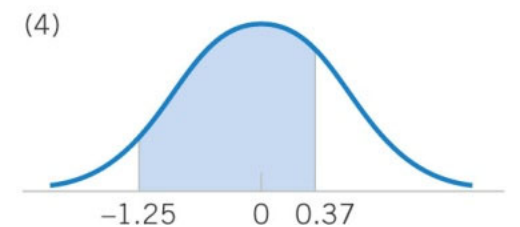
$$2. \quad P(Z < -0.86) = P(Z > 0.86) = 1 - P(Z < 0.86) = 1 - \textcolor{red}{0.815} = \textcolor{blue}{\underline{0.195}}$$



$$3. \quad P(Z > -1.37) = P(Z < 1.37) = \textcolor{red}{\underline{0.915}}$$



$$4. \quad P(-1.25 < Z < 0.37) = P(Z < 0.37) - P(Z < -1.25) \\ = P(Z < 0.37) - P(Z > 1.25) = P(Z < 0.37) - \\ (1 - P(Z < 1.25)) = \textcolor{red}{0.6443} - (1 - \textcolor{red}{0.8944}) = \textcolor{blue}{\underline{0.5387}}$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.523922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555670	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

Credit: XKCD
comics

WHY ARE THERE SLAVES IN THE BIBLE

WHY DO TWINS HAVE DIFFERENT FINGERPRINTS
WHY ARE AMERICANS AFRAID OF DRAGONS

WHY IS HTTPS CROSSED OUT IN RED
WHY IS THERE A LINE THROUGH HTTPS
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT

WHY AREN'T MY
ARMS GROWING



WHY ARE THERE WEEKS
WHY DO I FEEL DIZZY

QUESTIONS

FOUND IN GOOGLE AUTOCOMplete

WHY AREN'T ECONOMISTS RICH

WHY DO AMERICANS CALL IT SOCCER

WHY ARE MY EARS RINGING

WHY ARE THERE SO MANY AVENGERS

WHY ARE THE AVENGERS FIGHTING THE X MEN

WHY IS WOLVERINE NOT IN THE AVENGERS

WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN

WHY IS PSYCHIC WEAK TO BUG

WHY DO CHILDREN GET CANCER

WHY IS POSEIDON ANGRY WITH ODYSSEUS

WHY IS THERE ICE IN SPACE

WHY ARE THERE SWARMS OF GNATS

WHY IS THERE PHLEGM

WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN

WHY IS PSYCHIC WEAK TO BUG

WHY DO CHILDREN GET CANCER

WHY IS POSEIDON ANGRY WITH ODYSSEUS

WHY IS THERE ICE IN SPACE

WHY ARE THERE SWARMS OF GNATS

WHY IS THERE PHLEGM

WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN

WHY IS PSYCHIC WEAK TO BUG

WHY DO CHILDREN GET CANCER

WHY IS POSEIDON ANGRY WITH ODYSSEUS

WHY ARE THERE ANTS IN MY LAPTOP

WHY ARE THERE
GHOSTS



WHY IS THERE AN OWL IN MY BACKYARD

WHY IS THERE AN OWL OUTSIDE MY WINDOW

WHY IS THERE AN OWL ON THE DOLLAR BILL

WHY DO OWLS ATTACK PEOPLE

WHY ARE AK 47s SO EXPENSIVE

WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE

WHY ARE THERE GODS

WHY ARE THERE TWO SPOCKS

WHY IS MT VESUVIUS THERE

WHY DO THEY SAY T MINUS

WHY ARE THERE OBELISKS

WHY ARE WRESTLERS ALWAYS WET

WHY ARE OCEANS BECOMING MORE ACIDIC

WHY IS ARWEN DYING

WHY AREN'T MY QUAIL LAYING EGGS

WHY AREN'T MY QUAIL EGGS HATCHING

WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA

WHY ARE CIGARETTES LEGAL
WHY ARE THERE DUCKS IN MY POOL
WHY IS JESUS WHITE
WHY IS THERE LIQUID IN MY EAR
WHY DO Q TIPS FEEL GOOD
WHY DO GOOD PEOPLE DIE

WHY AREN'T
THERE GUNS IN
HARRY POTTER



WHY ARE ULTRASOUNDS IMPORTANT
WHY ARE ULTRASOUND MACHINES EXPENSIVE
WHY IS STEALING WRONG

WHY DO WHALES JUMP
WHY ARE WITCHES GREEN
WHY ARE THERE MIRRORS ABOVE BEDS
WHY DO I SAY UH
WHY IS SEA SALT BETTER

WHY ARE THERE TREES IN THE MIDDLE OF FIELDS
WHY IS THERE NOT A POKEMON MMO
WHY IS THERE LAUGHING IN TV SHOWS
WHY ARE THERE DOORS ON THE FREEWAY
WHY ARE THERE SO MANY SVCHOST.EXE RUNNING
WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA
WHY ARE THERE SCARY SOUNDS IN MINECRAFT
WHY IS THERE KICKING IN MY STOMACH
WHY ARE THERE TWO SLASHES AFTER HTTP

WHY ARE THERE CELEBRITIES
WHY DO SNAKES EXIST
WHY DO OYSTERS HAVE PEARLS
WHY ARE DUCKS CALLED DUCKS
WHY DO THEY CALL IT THE CLAP
WHY ARE KYLE AND CARTMAN FRIENDS
WHY IS THERE AN ARROW ON AANG'S HEAD
WHY ARE TEXT MESSAGES BLUE
WHY ARE THERE MUSTACHES ON CLOTHES
WHY ARE THERE MUSTACHES ON CARS
WHY ARE THERE MUSTACHES EVERYWHERE
WHY ARE THERE SO MANY BIRDS IN OHIO
WHY IS THERE SO MUCH RAIN IN OHIO
WHY IS OHIO WEATHER SO WEIRD

WHY ARE THERE MALE AND FEMALE BIKES
WHY ARE THERE BRIDESMAIDS
WHY DO DYING PEOPLE REACH UP
WHY AREN'T THERE VARICOSE ARTERIES
WHY ARE OLD KINGDOMS DIFFERENT

WHY ARE THERE TINY SPIDERS IN MY HOUSE
WHY DO SPIDERS COME INSIDE
WHY ARE THERE HUGE SPIDERS IN MY HOUSE
WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE
WHY ARE THERE SPIDERS IN MY ROOM
WHY ARE THERE SO MANY SPIDERS IN MY ROOM
WHY DO SPIDER BITES ITCH
WHY IS DYING SO SCARY

WHY IS THERE NO GPS IN LAPTOPS
WHY DO KNEES CLICK
WHY AREN'T THERE E GRADES
WHY IS ISOLATION BAD
WHY DO BOYS LIKE ME
WHY DON'T BOYS LIKE ME
WHY IS THERE ALWAYS A JAVA UPDATE
WHY ARE THERE RED DOTS ON MY THIGHS
WHY IS LYING GOOD

WHY IS THERE NO GPS IN LAPTOPS
WHY DO KNEES CLICK
WHY AREN'T THERE E GRADES
WHY IS ISOLATION BAD
WHY DO BOYS LIKE ME
WHY DON'T BOYS LIKE ME
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WHY AREN'T THERE E GRADES
WHY IS ISOLATION BAD
WHY DO BOYS LIKE ME
WHY DON'T BOYS LIKE ME
WHY IS THERE ALWAYS A JAVA UPDATE
WHY ARE THERE RED DOTS ON MY THIGHS
WHY IS LYING GOOD

WHY IS SEX
SO IMPORTANT

