Transformation Matrices

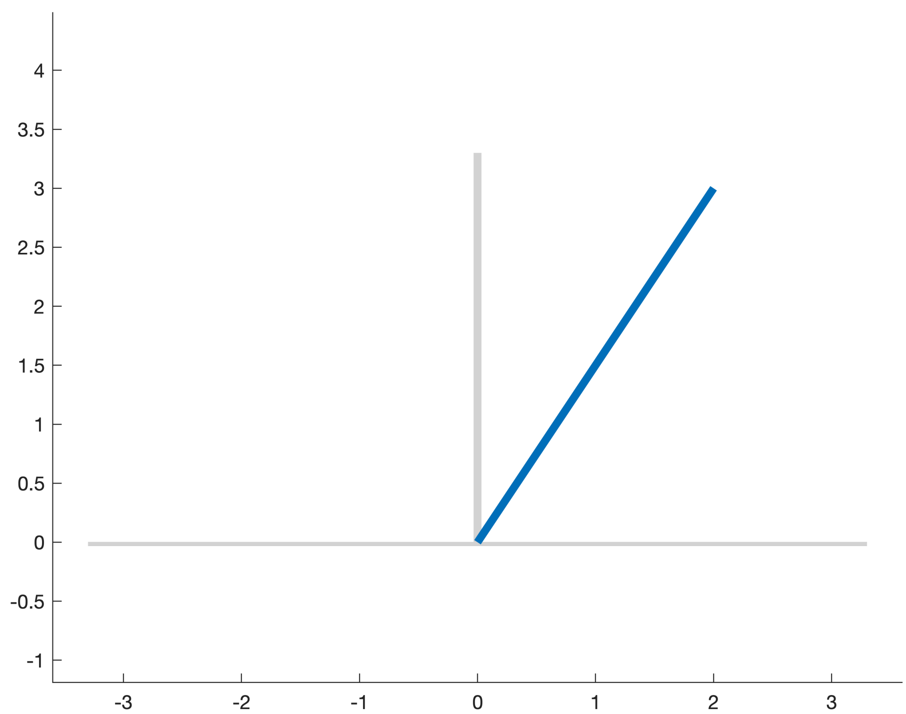
Rotation Matrices

To rotate a vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, we use a *rotation matrix:*

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

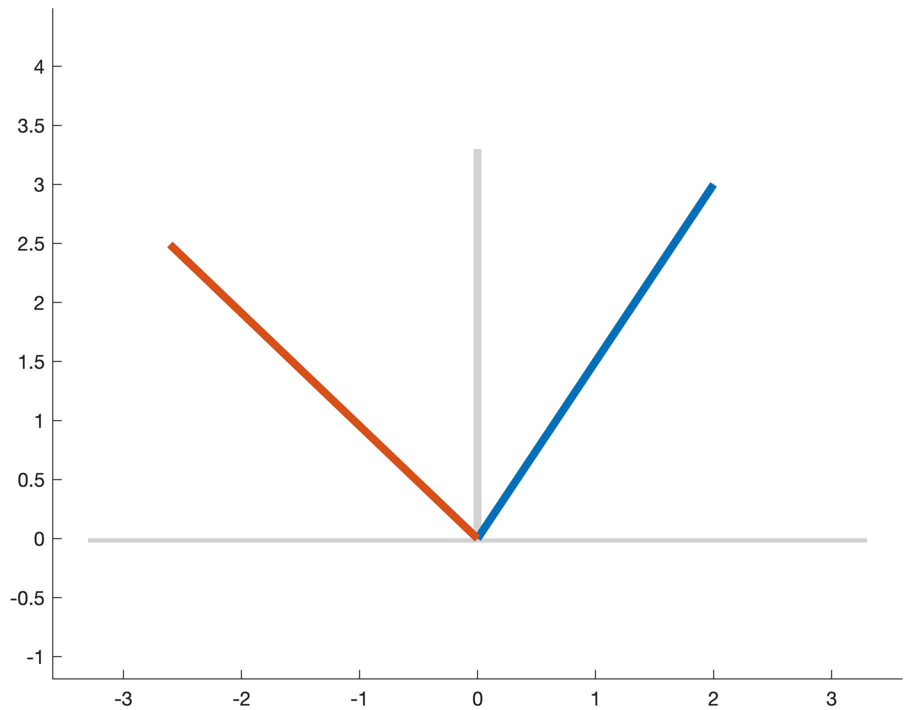
For example, consider the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

```
plotVectors([2;3])
```



Let's rotate this vector by 80 degrees (or $80/180\pi = 1.396$ radians).

```
R = [cos(1.396) -sin(1.396);
     sin(1.396)  cos(1.396)];
plotVectors([2;3], R*[2;3])
```



For convenience, Matlab has a function `deg2rad` that converts between degrees and radians. There's also a `rad2deg` to convert from radians to degrees.

```
deg2rad(80)
```

```
ans = 1.3963
```

```
rad2deg(pi)
```

```
ans = 180
```

Properties of Rotation Matrices

The columns of the rotation matrix are orthogonal:

$$\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \cdot \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix} = \cos\theta(-\sin\theta) + \sin\theta\cos\theta = 0$$

Also, the magnitude of each column is one (by a trigonometric identity):

$$(\cos\theta)^2 + (\sin\theta)^2 = (-\sin\theta)^2 + (\cos\theta)^2 = 1$$

If the rotation matrix has orthogonal columns and each column is a unit vector, then the rotation matrix is an orthogonal matrix. This implies that the inverse of a rotation matrix is simply its transpose.

$$R^{-1}(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}^{-1} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}^{T} = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$$

We've defined $R(\theta)$ to rotate in the counter-clockwise direction. The inverse $R^{-1}(\theta)$ must apply the same rotation in the clockwise direction since $R^{-1}(\theta)R(\theta)\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{x}$.

Translation Matrices

What if we want to *translate* a vector, i.e. move it by some distances Δx and Δy? Is there a matrix $T(\Delta x, \Delta y)$ such that $T\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}x + \Delta x \\ y + \Delta y\end{bmatrix}$?

The answer is no. Translation is an affine shift, so we cannot define it using a strictly linear operator (a matrix). However, we can perform translations in 2D space if we add a third "dummy" dimension. Let's write our 2D vector as a 3D vector by giving the third dimension the value 1:

$$\begin{bmatrix}x \\ y\end{bmatrix} \rightarrow \begin{bmatrix}x \\ y \\ 1\end{bmatrix}$$

Now we can define the translation matrix $T(\Delta x, \Delta y)$:

$$T(\Delta x, \Delta y) = \begin{bmatrix}1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1\end{bmatrix}$$

where

$$\begin{bmatrix}1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x \\ y \\ 1\end{bmatrix} = \begin{bmatrix}x + \Delta x \\ y + \Delta y \\ 1\end{bmatrix}$$

Notice how the translation matrix regenerates the 1 as the last entry. The third dimension allows us to perform affine transformations, but the entries in this dimension are never changed.

Combining Rotation and Transformation

We can use matrix multiplication to perform sequential matrix operations. First, we need to add a "dummy" dimension to our rotation matrices to make them compatible with our translation matrices:

$$R(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}$$

The one in the bottom right corner regenerates the one in the dummy dimension. The columns of our new rotation matrix are still orthogonal and have unit magnitude; thus the inverse of this matrix is still the transpose.

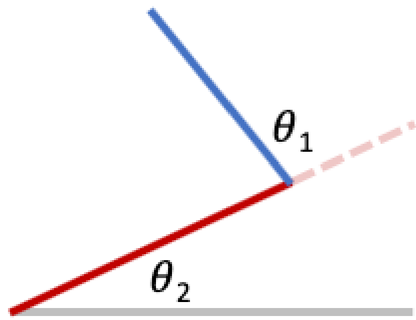
$$R^{-1}(\theta) = \begin{bmatrix}\cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}$$

To rotate a vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ by 0.3 radians and shift it by 2 in both dimensions, we apply two transformations:

$$T(2,2)R(0.3)\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos 0.3 & -\sin 0.3 & 0 \\ \sin 0.3 & \cos 0.3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.98 \\ 6.02 \\ 1 \end{bmatrix}$$

Analyzing Linkage Systems

A common problem in robotics is finding the position of the end of an arm with multiplie linkages (a series of rigid segments connected by angles). Let's analyze an arm with two segments and two flexible joints:



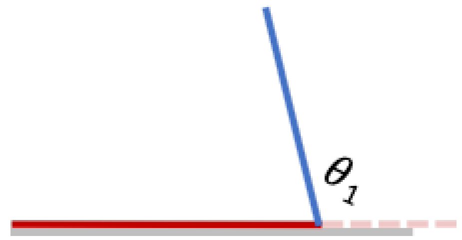
The first (blue) segment has length l_1. It is bent relative to the second segment at angle θ_1. (We always define this angle as a counter-clockwise rotation away from the previous segment, which we extended as a faint dashed line.) The second (red) segment has length l_2 and is rotated at angle θ_2 from the horizontal axis.

Given lengths l_1 and l_2 and angles θ_1 and θ_2, what is the position of the end of the blue segment? We can think of the arm as a series of transformations building up from the far end of the arm.

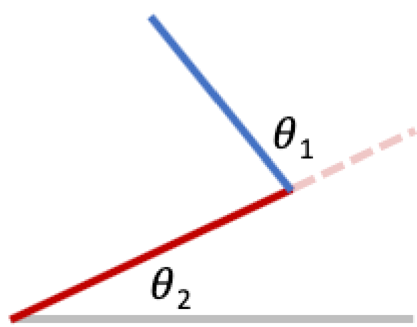
- The blue segment is a translation in the horizontal direction by length l_1

- The blue segment is rotated counter-clockwise by angle θ_1

- The red segment is a horizontal translation by length l_2



- The red segment is rotated counter-clockwise by angle θ_2



We apply each of these transformation in sequence, starting at the origin:

$$\mathbf{x}_{\text{end}} = R(\theta_2)T(l_2, 0)R(\theta_1)T(l_1, 0)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

For example, let's have $l_1 = l_2 = 3$, $\theta_1 = 30^\text{o}$, and $\theta_2 = 45^\text{o}$. Then the final position is

$$\mathbf{x}_{\text{end}} = R(45^\text{o})T(3, 0)R(30^\text{o})T(3, 0)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 2 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.259 & -0.966 & 2.90 \\ 0.966 & 0.259 & 5.02 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.90 \\ 5.02 \\ 1 \end{bmatrix}$$

The final position of the arm is at $x = 2.90$ and $y = 5.02$.