## Review

**Matrix Decompositions** (Matrix → product of matrices)

$$\underline{A} = \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1}$$ Eigendecomposition

$\underline{V}$ ← eigenvectors

$\underline{\Lambda}$ ← diagonal $(\lambda_1, \dots \lambda_n)$

$\underline{A}$ must be perfect (square, full set of $\lambda, \underline{v}$)

Mult. by $\underline{A}$

$\underline{x}$ → Eigenbasis → scale by Eigenvalues → back to starting basis

$\underline{V}^{-1}$ $\qquad$ $\underline{\Lambda}$ $\qquad$ $\underline{V}$

$$\underline{A}\,\underline{x} = \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1}\underline{x}$$

# Singular Value Decomposition (SVD)

$$\underline{A} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T$$

$m \times n \quad (m \times m)(m \times n)(n \times n)$

$\underline{U}, \underline{V}$ are unitary ($\underline{U}^{-1} = \underline{U}^T$)
orthonormal bases

$\underline{\Sigma}$ diagonal, $\sigma_i \geq 0$

$\underline{U}$: cols are "left singular vectors"

$\underline{V}$: cols are "right singular vectors"

$\underline{\Sigma}$: nonzero elements are singular values.

**NOTE:**

If $\underline{A}$ is perfect:

$$\underline{A} = \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1} \quad \text{(eigendecomp)}$$

$$\neq \quad \neq \quad \neq$$

$$\underline{A} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T \quad \text{(SVD)}$$

$\underline{V}$ from eigendecomp $\neq \underline{V}$ from SVD.

What is the relationship b/w EigenDecomp & SVD?

ed. $A = W \Lambda W^{-1}$     $W$: matrix of eigenvectors.

SVD $A = U \Sigma V^T$

Right singular vectors ($V$) are the eigenvectors of $A^T A$.

$A^T A$ called covariance matrix of $A$.

$$A^T A = (V \Sigma^T U^T)(U \Sigma V^T)$$
$$= V \Sigma^T U^T U \Sigma V^T$$
$$= V \Sigma^T \boxed{U^{-1} U} \Sigma V^T$$
$$= V \Sigma^T \Sigma V^T$$
$$= V \Sigma^2 V^{-1}$$

SVD of $A = U \Sigma V^T$

SVD of $A^T = (U \Sigma V^T)^T$

$= V \Sigma^T U^T$

$V, U$ are unitary

$U^T = U^{-1}$

$\Sigma$ is diagonal: $\Sigma^T = \Sigma$

$V$: are eigenvectors of $A^T A$

$\Sigma^2$ are the eigenvalues $\rightarrow \sigma_i = \sqrt{\lambda_i}$ of $A^T A$.

# What is SVD doing?

$$\underline{A} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T = \underline{U}\,\underline{\Sigma}\,\underline{V}^{-1}$$

$n > m$ $\quad \underline{A} =$ 

$\underline{x}$

$\downarrow$

$a_1 \underline{v}_1 + a_2 \underline{v}_2 \cdots + a_m \underline{v}_m + \cdots a_n \underline{v}_n$

$\downarrow$

scale by $\sigma_i$ along each basis

$\underline{\Sigma} =$  only first $m$ $\sigma_i$ are $\neq 0$

$\downarrow$

$\sigma_1 a_1 \underline{v}_1 + \cdots \sigma_m a_m \underline{v}_m + 0 a_{m+1} \underline{v}_{m+1} \cdots 0 a_n \underline{v}_n$

$\downarrow$

transfer to columns of $\underline{U}$

$\downarrow$

$\sigma_1 a_1 \underline{u}_1 + \sigma_2 a_2 \underline{u}_2 + \cdots + \sigma_m a_m \underline{u}_m$

$\underline{A} \in \mathbb{R}^{m \times n}$

$m > n$ $\quad \underline{A} =$ 

$\underline{x}$

$\downarrow$

$a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_n \underline{v}_n$

$\downarrow$

scale by sing. values

$\underline{\Sigma} =$  only $n$ $\sigma_i \neq 0$

$\downarrow$

$\sigma_1 a_1 \underline{v}_1 + \sigma_2 a_2 \underline{v}_2 + \cdots + \sigma_n a_n \underline{v}_n$

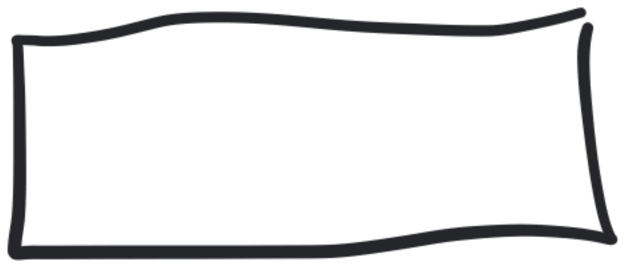
$\downarrow$

Transfer to columns of $\underline{U}$

$\downarrow$

$\sigma_1 a_1 \underline{u}_1 + \cdots + \sigma_n a_n \underline{u}_n + 0 \underline{u}_{n+1} + \cdots + 0 \underline{u}_m$

SVD "compressing" data into new composite variables.

$\underline{A} =$  $n > m$

$$\mathbb{R}^n \longrightarrow \boxed{\underline{A}} \longrightarrow \mathbb{R}^m$$

Output colums have decreasing information content.

$\underline{V}$ : right singular values. $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq \sigma_{m+1} = 0$

$\underline{v}_1 \longrightarrow \sigma_1 \longleftarrow$ largest; most information

$\underline{v}_2 \longrightarrow \sigma_2$

$\vdots$

$\underline{v}_m \longrightarrow \sigma_m \longleftarrow$ smallest (nonzero); least information

$\underline{v}_{m+1} \longrightarrow 0$

$\vdots$

$\underline{v}_n \longrightarrow 0$