BIOE 210

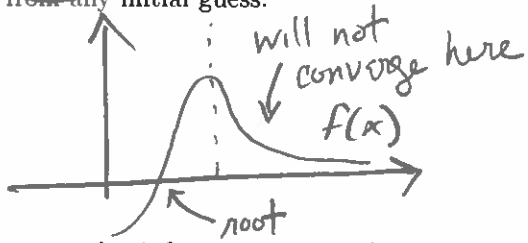
PRACTICE EXAM 3

You have 80 minutes to complete this exam.
You may use notes or printouts from the course website,
but **no electronic resources**.

Circle your final answer for each question.

PART I (20 POINTS; 5 POINTS EACH)

(1) TRUE or FALSE. If a nonlinear system has only one solution, Newton's method will converge ~~from any~~ initial guess.



(2) A genetic counselor informs a patient that their odds of getting disease are 1 in 300. What is the probability the patient will get the disease?

$$P(d) = \frac{odds(d)}{1+odds(d)} = \frac{1/300}{1+1/300} \approx 0.0033$$

(3) TRUE or FALSE. Every square matrix has a pseudoinverse.

only if $X^T X$ is full rank.

(4) TRUE or FALSE. It is necessary to have more observations than parameters to find a least-squares estimate using linear regression.

you can find an estimate; however you won't have confidence intervals on that estimate.

PART II (40 POINTS)

You are given a set of inputs ($\mathbf{x}$) and responses ($\mathbf{y}$)

x	y
0	2.1
1	3.1
2	5.8
3	11.3

You hypothesize that these data fit one of two models: $y = \beta_0 + \beta_1 x + \beta_2 x^2$ or $y = \beta_0 + \beta_1 e^x$. You want to fit the parameters using linear regression.

Construct a design matrix for each model using the data in the above table. (The design matrix is the matrix $\mathbf{X}$ in the linear system $\mathbf{y} = \mathbf{X}\beta + \epsilon$.)

for $y = \beta_0 + \beta_1 x + \beta_2 x^2$

$$\underline{X} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

for $y = \beta_0 + \beta_1 e^x$

$$\underline{X} = \begin{pmatrix} 1 & e^0 \\ 1 & e^1 \\ 1 & e^2 \\ 1 & e^3 \end{pmatrix}$$

You use `fitlm` to fit both models, with the following outputs.

```
Linear regression model:
    y ~ 1 + x + x^2

Estimated Coefficients:
                   Estimate       SE        tStat       pValue
                   --------    -------    -------    --------

    (Intercept)      2.155     0.23974     8.9889    0.070533
    x               -0.345       0.385    -0.8961     0.53485
    x^2              1.125     0.12298     9.1476    0.069319


Number of observations: 4, Error degrees of freedom: 1
Root Mean Squared Error: 0.246
R-squared: 0.999,  Adjusted R-Squared 0.996
F-statistic vs. constant model: 421, p-value = 0.0344
```

```
Linear regression model:
    y ~ 1 + expx

Estimated Coefficients:
                   Estimate       SE         tStat       pValue
                   --------    --------     ------    ---------

    (Intercept)     1.8553      0.25642     7.2353     0.018572
    expx           0.47699     0.023746     20.087    0.0024692


Number of observations: 4, Error degrees of freedom: 2
Root Mean Squared Error: 0.355
R-squared: 0.995,  Adjusted R-Squared 0.993
F-statistic vs. constant model: 403, p-value = 0.00247
```

Which model would give better predictions? Why?

The first (quadratic) model gives better predictions since the RMSE is lower (0.246 vs. 0.355).

Part III (40 points)

Find the Jacobian matrix for the following system of equations:

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_2 - 3x_1 \\ 2(x_1 + x_2) \end{pmatrix}$$

$$\underline{J}(\underline{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 2 & 2 \end{pmatrix}$$

Starting with an initial guess of $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find $\mathbf{x}_1$. Use $\mathbf{J}(\mathbf{x}_0)^{-1} = \begin{pmatrix} -1/4 & 1/8 \\ 1/4 & 3/8 \end{pmatrix}$.

$$\underline{x}_1 = \underline{x}_0 - \underline{J}(\underline{x}_0)^{-1}\, \underline{f}(\underline{x}_0)$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1/4 & 1/8 \\ 1/4 & 3/8 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Evaluate $\mathbf{f}(\mathbf{x}_1)$ to show that $\mathbf{x}_1$ is a root of $\mathbf{f}$.

$$\underline{f}(\underline{x}_1) = \begin{pmatrix} 0-3(0) \\ 2(0+0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $\underline{x}_1$ is a root of $\underline{f}$.

Why would Newton's method converge to a root after only one iteration?

$\underline{f}(\underline{x})$ is linear, so a first-order approximation is exact. The tangent plane points directly to the root.