BIOE 298, SECTIONS MFI & B

PRACTICE EXAM 1

You have 80 minutes to complete this exam.
You may use notes or printouts from the course website,
but **no electronic resources**.

Part I (40 points; 4 points each)

(1) True or False. The matrix $\begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ has an inverse.

must be square

2x3
Not square

(2) True or False. There exists a real number θ such that $\begin{pmatrix} 1 \\ \theta \\ 1/2 \end{pmatrix}$ is a unit vector.

$\|\underline{x}\| = 1$

$\left\|\begin{pmatrix} 1 \\ \theta \\ 1/2 \end{pmatrix}\right\| = \sqrt{1^2+\theta^2+(\tfrac{1}{2})^2} > 1 \Rightarrow$ Not a unit vector

(3) We said (many times) that the integers are not a field since they have additive inverses $(-a)$ for every element but not multiplicative inverses (a^{-1}). We can construct a set that contains both additive and multiplicative inverses using the integers by collecting 2^i and -2^i for every integer i:

$$\{\ldots, \pm 2^{-2}, \pm 2^{-1}, \pm 2^0, \pm 2^1, \pm 2^2, \ldots\}$$

Is this set a field?

$\nexists\, 0 \Rightarrow$ not a field

(4) $\|\mathbf{x}\| = 8$. What is $\|-3\mathbf{x}\|$?

$\|k\underline{x}\| = |k|\,\|\underline{x}\| \Rightarrow \|-3\underline{x}\| = |-3|\,\|\underline{x}\|$

$3 \times 8 = 24$

(5) Let $\begin{pmatrix} 0 & 1 & -2 \\ 0 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 12 \end{pmatrix}$. What is x_2?

$-x_2 = -8 \Rightarrow x_2 = 8$

(6) TRUE or FALSE. If the angle between $\mathbf{x} = \begin{pmatrix} 2a \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 4 \\ a \\ 2 \end{pmatrix}$ is 135°, then $\mathbf{x} \cdot \mathbf{y} = 7$.

$\underline{x} \cdot \underline{y} = \|\underline{x}\| \, \|\underline{y}\| \cos\theta$

$\theta = 135^\circ$

$\cos 135^\circ < 0$

(7) Which vectors are orthogonal to $\begin{pmatrix} 4 \\ 0 \\ 2 \\ 0 \end{pmatrix}$?

$x_1 y_1 + \cdots + x_n y_n$

(a) $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1/4 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ (d) $\begin{pmatrix} 0 \\ -12 \\ 0 \\ 8 \end{pmatrix}$

(8) TRUE or FALSE. $\mathbf{AB} \neq \mathbf{BA}$ for all matrices $\mathbf{A}$ and $\mathbf{B}$, even if $\mathbf{A}$ and $\mathbf{B}$ are conformable.

$\underline{B} = \underline{I} \Rightarrow \underline{A}\,\underline{B} = \underline{A}\,\underline{I} = \underline{I}\,\underline{A} = \underline{B}\,\underline{A}$

(9) Which of the following differential equations are linear in $\underline{u}$

(a)
$$\frac{\partial^2 u}{\partial x \partial y} + \underline{\sin(xy)}\, u = 4 \quad \checkmark$$

(b)
$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) = 0 \quad \checkmark$$

(c)
$$\underline{\frac{d^2 u}{dx^2}} + 3e^u x \frac{du}{dx} + u = 1 \quad \times$$

(d)
$$\frac{du}{dt} = tu \Leftarrow \frac{1}{u}\frac{du}{dt} = t \quad \checkmark$$

(10) What is the rank of the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$?

rank = 3

Part II (30 points)

Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

$(\underline{A}\ \underline{I}) \longrightarrow (\underline{I}\ \underline{A}^{-1})$

$\begin{pmatrix} 3 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$

$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 3 & -1 & 1 & 0 \end{pmatrix}$

$\xrightarrow{\frac{1}{2} R_1} \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 3 & -1 & 1 & 0 \end{pmatrix}$

$\xrightarrow{R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 1 & -3/2 \end{pmatrix}$

$\xrightarrow{-R_2} \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 3/2 \end{pmatrix}$

$\underline{A}^{-1} = \begin{pmatrix} 0 & 1/2 \\ -1 & 3/2 \end{pmatrix}$

Check: $\underline{A}\,\underline{A}^{-1} = \underline{I}$

$\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 \\ -1 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

Use the inverse to solve $\mathbf{Ax} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and $\mathbf{Ax} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\underline{x} = \underline{A}^{-1} \underline{y}$

For $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} 0 & 1/2 \\ -1 & 3/2 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} -5/2 \\ -19/2 \end{pmatrix}$

For $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} 0 & 1/2 \\ -1 & 3/2 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 11/2 \end{pmatrix}$

Part III (30 points)

Write equations for the finite difference approximation for the following ODE at **four nodes spanning [0, 3]**.

$$\frac{d^2u}{dx^2} - 4u = x^2, \quad u(0) = 1,\ u(3) = 4$$

2nd

$x = 0, 1, 2, 3 \Rightarrow \Delta x = 1$

$k = 0, 1, 2, 3$

@ node 1 $\quad \dfrac{u^{(0)} - 2u^{(1)} + u^{(2)}}{1^2} - 4u^{(1)} = 1^2$

$$u^{(0)} - 6u^{(1)} + u^{(2)} = 1$$

@ node 2 $\quad \dfrac{u^{(1)} - 2u^{(2)} + u^{(3)}}{1^2} - 4u^{(2)} = 2^2$

$$u^{(1)} - 6u^{(2)} + u^{(3)} = 4$$

@ node 0

$$u^{(0)} = 1$$

@ node 3 $\quad u^{(3)} = 4$

Rewrite the equations as a matrix equation of the form $\mathbf{Ax} = \mathbf{y}$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -6 & 1 & 0 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^{(0)} \\ u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \\ 4 \end{pmatrix}$$