TAM 211 - Worksheet 15

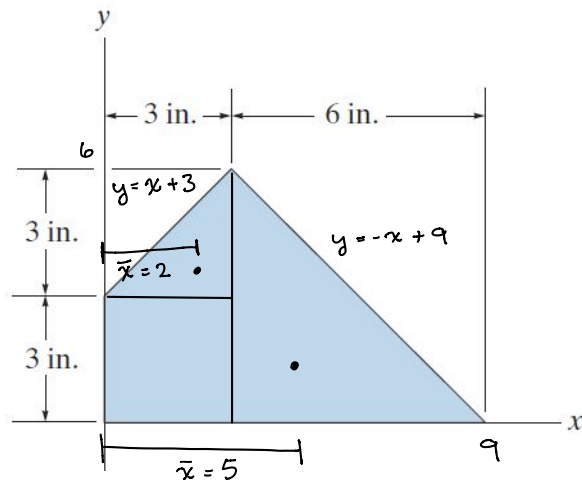## Moment of Inertia

1) Determine the moment of inertia of the composite area about the y-axis.

$$I_y = \int r^2 \, dA$$

$$I_y = \int_0^3 x^2 (x+3) \, dx + \int_3^9 x^2 (-x+9) \, dx$$

$$= \int_0^3 x^3 + 3x^2 \, dx + \int_3^9 -x^3 + 9x^2 \, dx$$

$$= \frac{1}{4} x^4 + x^3 \Big|_0^3 + -\frac{1}{4} x^4 + 3x^3 \Big|_3^9$$

$$\boxed{I_y = 533.25 \text{ in}^4}$$

Alternatively:

△ h $\overline{I}_y = \frac{1}{36} bh^3$
b

□ h $\overline{I}_y = \frac{1}{12} hb^3$
b

$$I_y = \overline{I}_y + A(dx)^2$$

$$= \left[\frac{1}{36}(3)(3)^3 + \frac{1}{2}(3)(3)(2)^2 \right] + \left[\frac{1}{12}(3)(3)^3 + 3(3)\left(\frac{3}{2}\right)^2 \right]$$

$$+ \left[\frac{1}{36}(6)(6)^3 + \frac{1}{2}(6)(6)(5)^2 \right]$$

$$\boxed{I_y = 533.25 \text{ in}^4}$$

2) Determine the location $\bar{y}$ of the centroid of the beam's cross-sectional area, then find the moment of inertia about the x'-axis that passes through its centroid. Neglect the size of the corner welds at A and B for the calculation, and assume the cross section can be approximated by two circles connected to a rectangle.
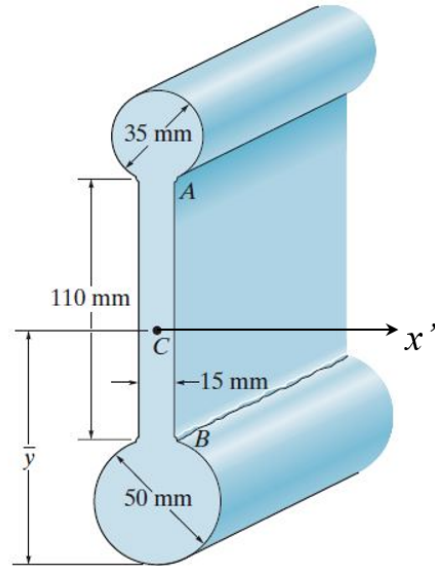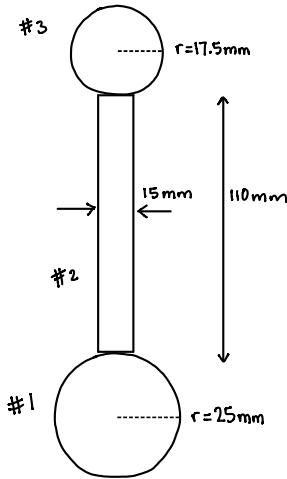
$\bar{y}_1 = 25\text{mm} \quad \bar{y}_2 = 105\text{mm} \quad \bar{y}_3 = 177.5\text{mm} \quad \bar{y} = \dfrac{\Sigma y_i A_i}{\Sigma A_i}$

Centroid: $\bar{y} = \dfrac{25(\pi)(25)^2 + 105(15)(110) + 177.5(\pi)(17.5)^2}{\pi(25)^2 + 110(15) + \pi(17.5)^2}$

$$\boxed{\bar{y} = 85.91478657 \text{ mm}}$$

Moment of Inertia:

$I_{x_1} = \frac{\pi}{4}(25)^4 + \left(\pi(25)^2\left(25 - \bar{y}\right)^2\right)$

$I_{x_2} = \frac{1}{12}(15)(110)^3 + \left(110(15)\left(105 - \bar{y}\right)^2\right)$

$I_{x_3} = \frac{\pi}{4}(17.5)^4 + \left(\pi(17.5)^2\left(177.5 - \bar{y}\right)^2\right)$

$I_{x'} = I_{x_1} + I_{x_2} + I_{x_3}$

$I_{x'} = 7592564.256 + 2264754.863 + 8143720.458$

$$\boxed{I_{x'} = 18001039.58 \text{ mm}^4}$$

Virtual Work

The principle of virtual work is useful for finding forces in static systems. For a system at equilibrium, the sum of the virtual work, dU, done by all the external forces must sum up to zero:

$$\sum dU = 0 \tag{1}$$

The equation to find the virtual work done by a force is

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{2}$$

where $\mathbf{F}$ is the force vector, and $d\mathbf{r}$ is the virtual displacement of the location where the force is applied. Virtual displacement is calculated by differentiating the position vector of the parameterized position vector (e.g. as a function of θ). For example, if $\mathbf{r} = x(\theta)\hat{\imath} + y(\theta)\hat{\jmath}$,
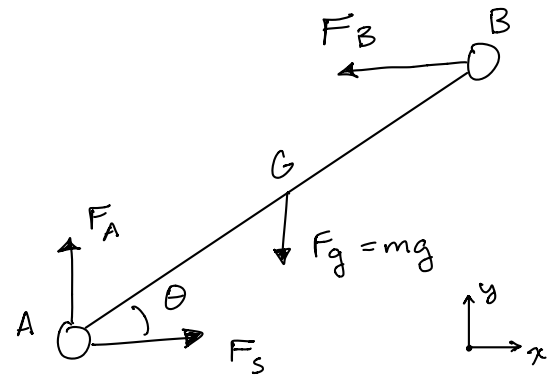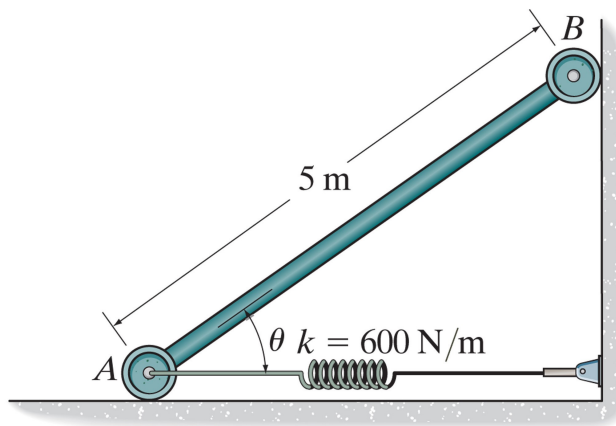
$$\frac{d\mathbf{r}}{d\theta} = \frac{dx}{d\theta}\hat{\imath} + \frac{dy}{d\theta}\hat{\jmath} \tag{3}$$

solving for $d\mathbf{r}$,

$$d\mathbf{r} = \left(\frac{dx}{d\theta}\hat{\imath} + \frac{dy}{d\theta}\hat{\jmath}\right) d\theta \tag{4}$$

3) The bar-spring system below is at resting at equilibrium. The mass of the bar is 50-kg, and the spring is unstretched when $\theta = 60^o$.

a) Draw a FBD for the bar and identify all the external forces.

b) Define the position vectors of the points where forces are applied from the origin at the bottom right corner of the figure as functions of θ. For example, $\mathbf{r}_A = -5\cos(\theta)\hat{\imath}$ m. Now find $\mathbf{r}_B$ and $\mathbf{r}_G$ (G is the center of gravity of the bar).

$$\vec{r}_B = 5\sin\theta \; \hat{\jmath} \; \text{m}$$

$$\vec{r}_G = -2.5\cos\theta\,\hat{\imath} + 2.5\sin\theta\,\hat{\jmath} \; \text{m}$$

3

c) Obtain the virtual displacements of each position vector from part (b) ($d\mathbf{r}_A$, $d\mathbf{r}_B$, and $d\mathbf{r}_G$) using Equation (4).

$$d\vec{r}_A = 5\sin\theta \, d\theta \, \hat{\imath}$$

$$d\vec{r}_G = 2.5\sin\theta \, d\theta \, \hat{\imath} + 2.5\cos\theta \, d\theta \, \hat{\jmath}$$

$$d\vec{r}_B = 5\cos\theta \, d\theta \, \hat{\jmath}$$

d) Express every force in the FBD from part (a) in vector form.

$$\vec{F}_A = [0, F_A] \, N$$

$$\vec{F}_g = [0, -mg] \, N$$

$$\vec{F}_B = [-F_B, 0] \, N$$

$$\vec{F}_s = [kx, 0] = [k(\overbrace{5\cos\theta - 2.5}^{x}), 0] \, N$$

unstretched spring length $= 5\cos(60°) = 2.5\,m$

e) Find the virtual work done by each external force on the bar using Equation (2) and parts (c) and (d).

$$F_A: \quad \vec{F}_A \cdot \delta\vec{r}_A = 0$$

$$F_B: \quad \vec{F}_B \cdot \delta\vec{r}_B = 0$$

no virtual work from forces F_A and F_B due to dot product $= 0$

$$F_s: \quad \vec{F}_s \cdot \delta\vec{r}_A = 25k\cos\theta\sin\theta \, \delta\theta - 12.5k\sin\theta \, \delta\theta$$

$$F_g: \quad \vec{F}_g \cdot \delta\vec{r}_G = -2.5mg\cos\theta \, \delta\theta$$

f) Use Equation (1) and part (d) to determine the angle of equilibrium, θ.

$$\sum \delta U = \sum \vec{F} \cdot \delta\vec{r} = 0$$

$$25k\cos\theta\sin\theta \, \cancel{\delta\theta} - 12.5k\sin\theta \, \cancel{\delta\theta} - 2.5mg\cos\theta \, \cancel{\delta\theta} = 0$$
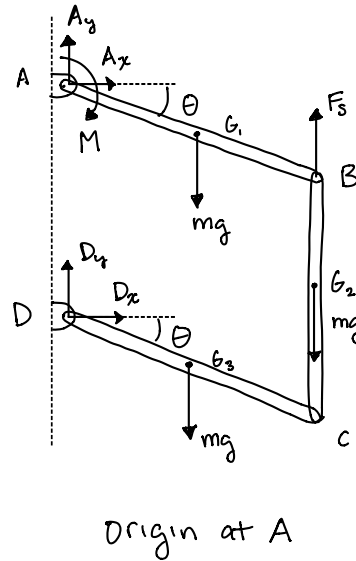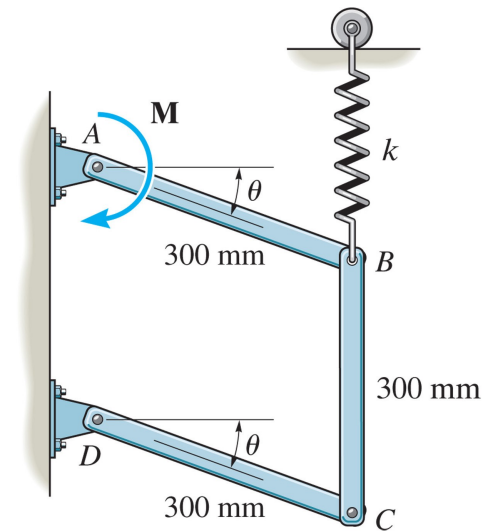
$$15000\cos\theta\sin\theta - 7500\sin\theta - 1226.25\cos\theta = 0$$

$$\theta \approx 9.545348613°$$

4) Each of the three links in the mechanism below have a mass of 5-kg. The spring has a stiffness constant of $k = 2.5$ kN/m, and it is unstretched when $\theta = 0°$. The moment $\mathbf{M}$ applied on the top link at A is 15 N·m.

$$K = 2500 \frac{N}{m} \qquad M = 15 Nm$$

a) Draw a FBD for the 3-links mechanism and identify all the forces and moments that will do work.

A_x, A_y, D_x, D_y will contribute no virtual work
(dot product $\vec{F} \cdot \delta \vec{r} = 0$)

Origin at A

b) Find the position coordinates of the forces and moments that will do work, measured from A.

$$\vec{\theta}_M = [0, 0, -\theta]$$
$$\vec{r}_B = [0.3\cos\theta, -0.3\sin\theta] m$$
$$\vec{r}_{G_1} = [0.15\cos\theta, -0.15\sin\theta] m$$

$$\vec{r}_{G_2} = [0.3\cos\theta, -0.3\sin\theta - 0.15] m$$
$$\vec{r}_{G_3} = [0.15\cos\theta, -0.15\sin\theta - 0.3] m$$

c) Find the virtual displacement of each of the forces and moments that will do work.

$$\delta\vec{\theta}_M = [0, 0, -1]\delta\theta$$
$$\delta\vec{r}_B = [-0.3\sin\theta, -0.3\cos\theta]\delta\theta$$
$$\delta\vec{r}_{G_1} = [-0.15\sin\theta, -0.15\cos\theta]\delta\theta$$
$$\delta\vec{r}_{G_2} = [-0.3\sin\theta, -0.3\cos\theta]\delta\theta$$
$$\delta\vec{r}_{G_3} = [-0.15\sin\theta, -0.15\cos\theta]\delta\theta$$

$$\vec{M} = [0, 0, -15] Nm$$
$$\vec{F}_s = [0, 0.3k\sin\theta] N$$
$$= [0, 750\sin\theta] N$$
$$\vec{F}_g = [0, -mg]$$
$$= [0, -49.05] N$$

dot product
$\Rightarrow$
to find virtual work

$$\vec{M} \cdot \delta\vec{\theta}_M = 15\,\delta\theta$$
$$\vec{F}_s \cdot \delta\vec{r}_B = -225\cos\theta\sin\theta\,\delta\theta$$
$$\vec{F}_g \cdot \delta\vec{r}_{G_1} = 7.3575\cos\theta\,\delta\theta$$
$$\vec{F}_g \cdot \delta\vec{r}_{G_2} = 14.715\cos\theta\,\delta\theta$$
$$\vec{F}_g \cdot \delta\vec{r}_{G_3} = 7.3575\cos\theta\,\delta\theta$$

d) Use the principle of virtual work to determine the angle of equilibrium, θ.

$$\sum\delta U = \sum\vec{F}\cdot\delta\vec{r} + \sum\vec{M}\cdot\delta\vec{\theta} = 0$$

$$0 = 15\,\delta\theta - 225\cos\theta\sin\theta\,\delta\theta + 7.3575\cos\theta\,\delta\theta + 14.715\cos\theta\,\delta\theta + 7.3575\cos\theta\,\delta\theta$$

$$0 = 15 - 225\cos\theta\sin\theta + 29.43\cos\theta$$

$$\longrightarrow \boxed{\theta \approx 11.468243364°}$$