

Lecture 6

• Closing some loopholes

- Schur's Lemma: two irreps ρ_1, ρ_2

a map A s.t. $\forall g \in G \quad A\rho_1(g) = \rho_2(g)A$

$\text{Ker}(A) = \{v \in V_1 : Av = 0\}$ is an
invariant subspace of ρ_1 and so is
either $= \{\emptyset\}$ or $= V_1$

We can consider $\text{Im}(A) = \{v \in V_2 \text{ s.t. } v = Aw \text{ for } w \in V_1\}$

$$= AV_1$$

$$f \in V \ominus \text{Im } A, \quad v = Aw$$

$$\Rightarrow \rho_2(g)v = \rho_2(g)Aw = A[\rho_1(g)w]$$

$$\Rightarrow \rho_2(g)v \in \text{Im } A$$

$\Rightarrow \text{Im}(A)$ is an invariant subspace of ρ_2

$$\Rightarrow \text{Im}(A) \subseteq \mathcal{O} \text{ or } \mathcal{O}^\perp$$

$\Rightarrow A$ is either 0 or invertible

Corollary 2 Suppose A is invertible

$$A\varrho_1(g) = \varrho_2(g) A$$

$$\Rightarrow \varrho_2(g) = A\varrho_1(g)A^{-1}$$

We can also consider $A^+ : V_2 \rightarrow V_1$

$$\hookrightarrow \varrho_1(g)^+ A^+ = A^+ \varrho_2^+(g)$$

$$\rightarrow \varrho_1(g^{-1}) A^+ = A^+ \varrho_2(g^{-1}) \quad (A^+ \text{ is invertible})$$

$$A^t A : V_1 \rightarrow V_1$$

$$A^t A \rho_1(s) = A^t \rho_2(s) A = \rho_1(s) A^t A$$

$$[A^t A, \rho_1(s)] = 0$$

Schw's lemma says $A^t A = \lambda \text{Id}$

which means $A^{-1} = \frac{1}{\lambda} A^t$

$$U^t U = I$$

$$U = \frac{1}{\sqrt{\lambda}} A$$

$$\begin{aligned} U^t U \\ = \frac{1}{\lambda} A^t A \\ = I \end{aligned}$$

$$\begin{aligned}\rho_2(g) &= A \rho_1(g) A^{-1} \\ &= U \rho_1(g) U^+ \Rightarrow \rho_1 \text{ and } \rho_2 \\ \text{are the same representation}\end{aligned}$$

$$\begin{aligned}&\langle \rho(g)x, \rho(g)y \rangle \\ &= \langle x, y \rangle\end{aligned}$$

$$\left[E_G^{(i_2 j_1)} \right]_{k l} = \sum_{g \in G} \left[e_2(g^{-1}) \right]_{k i} \left[e_1(g) \right]_{j_1 l}$$

so either:

$$e_2 \neq e_1 \quad \text{not unitarily equivalent}$$

$$\left[E_G^{(i_2 j_1)} \right]_{k l} = 0 \quad \forall i_2, j_1, k, l$$

so in particular

$$\sum_{i \in U_1} [E^{i_2 j_1}]_{i_1 j_1} = \sum_{g \in G} \sum_{i \in U_1} [e_g g^{-1}]_{i_1 i_2} [e_g]_{j_1 j_2}$$

$$= \sum_{g \in G} \chi_{e_2}(g^{-1}) \chi_{e_1}(g)$$

$$= \sum_{g \in G} \chi_{e_2}^*(g) \chi_{e_1}(g)$$

so we can define

$$\langle \chi_{e_2}, \chi_{e_1} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{e_2}^*(g) \chi_{e_1}(g)$$

β if e_1 and e_2
are inequivalent

$$\text{if } e_1 = e_2$$

,

$$[E_G^{ij}]_{ke} = \sum_{g \in G} [e_1(g^{-1})]_{kir} [e_1(g)]_{j,l} = \lambda_{ij,k,l}^{G|S}$$

$$\sum_k [E_G^{ij}]_{kk} = \sum_{g \in G} [e_1(g^{-1})]_{kir} [e_1(g)]_{j,k} = |G| \lambda_{ij,1}^{\dim e_1}$$

$$= \sum_{g \in G} [e_i(s) R_i(g^{-1})]_{j_1 j_2}$$

$$= \sum_{g \in G} [e_i(E)]_{j_1 j_2} = \sum_{g \in G} S_{i_2 j_1} = 16/S_{i_2 j_1}$$

$$\Rightarrow \lambda_{i_2 j_1} = S_{i_2 j_1} / \dim P_1$$

$$\langle X_e, X_{e_1} \rangle = \frac{1}{16k} \sum_{i_2 j_1} [E_6^{i_2 j_1}]_{j_1 j_2} = \frac{16}{16k} \sum_{i_2 j_1} S_{i_2 j_1} S_{i_2 j_1} = \frac{\sum S_{i_2 j_1}}{\dim P_1}$$

$$= \sum_{i \in I} \frac{s_{i,n}}{\dim e_i} = 1$$

The upshot: if ρ_1 and ρ_2 are irreducible representations

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \begin{cases} 0 & \text{if they are different} \\ 1 & \text{if they are the same} \end{cases}$$

$$\sum_{g \in G} [\rho_2(g^{-1})]_{ij} [\rho_1(g)]_{kl} = \begin{cases} 0 & \text{if } \rho_1 \neq \rho_2 \\ \frac{|G|}{\dim \rho_1} \delta_{il} \delta_{jk} & \rho_1 = \rho_2 \end{cases}$$

The irreducible characters form an orthonormal basis for the characters of all representations

i.e. if we have

$$e = e_1 \oplus e_1 \oplus e_2 \oplus e_3 \oplus \dots$$

$$x_e = x_{e_1} + x_{e_1} + x_{e_2} + x_{e_3}$$

$$\rightarrow \langle x_{e_i}, x_e \rangle = (\text{the # of times } e_i \text{ appears in the decomposition of } e)$$

= multiplicity of e_i in e

\rightarrow You will often see representations like

in the form of character tables

the # of irreps
of a grp = the
of conjugacy classes
[character tables are
square matrices]

$$\chi(g) = \chi(hgh^{-1})$$

I can consider a vector space
of fns called class functions

$f: G \rightarrow \mathbb{C}$ that's constant
on conjugacy classes

→ the irreducible characters form
an orthonormal basis for the space of
class functions

Ex: Let's look at the point
group $G = 2mm$
 $= \{E, C_{2z}, M_x, M_y\}$

This group is Abelian

$$g_1 g_2 = g_2 g_1$$

$$M_X M_Y = C_{23} = M_Y M_X$$

$$C_{23} M_X = M_X C_{23} = M_Y$$

$$M_X^2 = M_Y^2 = C_{23}^2 = E$$

	E	C_{23}	M_Y	M_X
A_1	1	1	1	1
A_2	1	1	-1	-1
B_1	1	-1	1	-1
B_2	1	-1	-1	1

$$\textcircled{1} \quad \chi_\rho(E) = \text{tr}[\rho(E)]$$

pf: Schur's lemma. Suppose we have f that's not a linear comb. of irreducible characters

$$\rightarrow \langle f, \chi_{c_i} \rangle = 0 \quad \forall c_i$$

$$f_i = \sum_{g \in G} f(g^{-1}) \rho_i(g) = \lambda_i \cdot \text{Id}$$

by Schur's lemma

$$\lambda = \frac{\text{tr } f_i}{\dim \rho_i} = 0$$

$\Rightarrow f_i = 0$ if representations

$$= \text{tr}[\text{Id}]$$

$$= \dim \mathbb{C}$$

Since $\text{tr}[1 \times 1 \text{ matrix}] = 1 \times 1 \text{ matrix}$

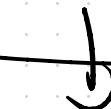
ID representations are equal to characters

$$A_i: \rho_{A_i}(g) = 1 \forall g$$

Let's look at a reducible representation

so we can consider the regular representation

$$\rightarrow f = 0$$



Serre's book,

Fulton & Harris "Representation Theory: A first course"

given a class function f

$$f(g) = \sum_i \langle f, \chi_{P_i} \rangle \chi_{P_i}(g)$$

where $\langle f, \chi_{P_i} \rangle = \frac{1}{|G|} \sum_{g \in G} f(g^{-1}) \chi_{P_i}(g)$

$$V = \begin{pmatrix} x \\ y \end{pmatrix}$$

Vector representation

$$e_V(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e_V(C_{23}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$e_V(M_x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e_V(M_y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$x_{P_1} + \sqrt{\pi} i x_{P_2}$$

$$\begin{array}{c|ccccc} & E & C_{23} & M_x & M_y \\ \hline x_V & 2 & -2 & 0 & 0 \end{array}$$

$$\begin{array}{c|ccccc} & E & C_{23} & M_y & M_x \\ \hline A_1 & 1 & 1 & 1 & 1 \\ A_2 & 1 & 1 & -1 & -1 \\ B_1 & 1 & -1 & 1 & -1 \\ B_2 & 1 & -1 & -1 & 1 \end{array}$$

$$\langle \chi_V, \chi_{A_1} \rangle = \frac{1}{4} (2, -2, 0, 0) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4} (2 - 2 + 0 + 0) = 0$$

$$\langle \chi_V, \chi_{A_2} \rangle = \frac{1}{4} (2, -2, 0, 0) \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$\langle \chi_V, \chi_{B_1} \rangle = \frac{1}{4} (2, -2, 0, 0) \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{4} (2 - -2) = 1$$

$$\langle \chi_V, \chi_{B_2} \rangle = 1$$

$$\chi_V = \chi_{B_1} + \chi_{B_2} \rightarrow P_V = P_{B_1} \oplus P_{B_2}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\psi_{P_X}^{(x,y,z)} = R(r) \sin\theta \cos\phi = R(r) \frac{x}{r}$$

$$\left\{ \begin{array}{l} C_{2z}: \Psi_{\rho_x}(-x, -y, z) = R(r) \frac{1}{r} (-x) = -\Psi_{\rho_x}(x, y, z) \\ M_x: \Psi_{\rho_x}(-x, y, z) = -\Psi_{\rho_x}(x, y, z) \\ M_y: \Psi_{\rho_x}(x, -y, z) = \Psi_{\rho_x}(x, y, z) \end{array} \right.$$

$$e_{\rho_x}(C_{2z}) = -1 \quad e_{\rho_x}(M_y) = +1 \quad \text{B1}$$

$$e_{\rho_x}(M_x) = -1 \quad e_{\rho_x}(E) = +1$$