Given, a set of orbitals at Wyckoff position $\Gamma_n$ transform in irreps $\mathbf{e}$ of $G_{\Gamma_n}$

$$\rightarrow \langle \psi_\beta \theta_l \mid U_{\beta l} \psi_{\alpha k} \rangle = e^{i \theta_k} t_{\alpha \beta} \epsilon_{\alpha \beta \gamma} (h_{\alpha \beta}) = \beta^k_{\alpha \beta \gamma} \langle \gamma \rangle$$

$$g = t_{\alpha \beta} \; g_{\alpha} \; h_{\alpha \beta} \; g_{\beta}^{-1} \quad \text{for coset representatives} \; g_{\alpha}, \; g_{\beta}, \; h_{\alpha \beta} \in G_{\Gamma_n}$$
Example: Construct EBRs for a 1D inversion-symmetric chain:

$G_{\text{1a}} = \langle \{1|0\} \rangle$

$\{E, I\} = T$

$G = T \times G_{\text{1a}}$ only one coset.
\[ g_\alpha = \{ E_{103} \} \]
\[ \{ I | 03 \} = \{ E_{103} \} \{ E_{103} \} \{ E_{103} \} \{ E_{103} \} \]
\[ e_\alpha \xi = g_\alpha \cdot g_\beta \cdot g_\alpha^{-1} \]
\[ B_{1_\alpha}^k (\{ I | 03 \}) = \left\{ \begin{array}{l}
    e_A (F_{103}) \cdot (A_{1_\alpha})^\Gamma \in G \\
    e_B (F_{103}) \cdot (B_{1_\alpha})^\Gamma \in G
  \end{array} \right. \]

In this case, \( B_{1_\alpha}^k (\{ I | 03 \}) \) is \( k \)-independent.
\[(A_{i0})^{\gamma_6} + 1\]

\[(B_{i0})^{\gamma_6} - 1\]

\[k = 0\ vs\ k = \overline{11}\]

\[A_{i0}^{\gamma_6} + 1\]

\[B_{i0}^{\gamma_6} - 1\]
What about the 1B position:

\[ 1b: \left( \frac{a}{2} \right) = \overline{r}_b \quad \{ I | a \} \overline{r}_b = \overline{r}_b \]

\[ G_{1b} = \left\{ \{ E1 \overline{0}\}, \{ I | a \} \right\} \subseteq T \]

\begin{array}{c|cc}
\text{Irreps} & E & I \\
\hline
\overline{p}_A & +1 & +1 \\
\overline{p}_B & +1 & -1 \\
\end{array}

\[ \{ I | \overline{0}\} = \left\{ E1 | -a \right\} \{ I | a \} \\
^t t'^a G_{1b} \]
\[
B_{1b}^k({\{1\|0\}}) = e^{-i(k) \cdot (-a)} \begin{cases} e^A({\{1\|a\}}) & A \\ e^{-i(ka)} & B \\ e^{-i(ka)} & \end{cases} 
\]

<table>
<thead>
<tr>
<th></th>
<th>k=0</th>
<th>k=\pm J/\hbar</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_{1b})_{1G}</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>(B_{1b})_{1G}</td>
<td>-1</td>
<td>+1</td>
</tr>
</tbody>
</table>
Recall: For the 1d chain
\[ h(k) = (\Delta + t_1 \cos k a) \sigma_2 + t_2 \sin k a \sigma_3 \]

\[ \Delta > t_1 > 0 \]

WFs localized at \( \overline{\Gamma} = 0 \)

\( (\beta_{1a}) \bar{g} \)

\[ t_1 > \Delta > 0 \]

WFs localized at \( \overline{\Gamma} = \frac{9}{2} \)

\( (\beta_{1b}) \bar{g} \)

What about all bands?
\[ S + \rho \otimes \Gamma = 0 \]

\[ (\alpha_{1a})_{IG} \quad (\beta_{1a})_{IG} \]

both bands together: \( A_{1a}^{\uparrow IG} \oplus B_{1a}^{\uparrow IG} \equiv A_{1b}^{\uparrow IG} \oplus B_{1b}^{\uparrow IG} \]

\[ \gamma_{2c}^\uparrow IG \]

\[ G_{2c} = \{ \{ E \in 10 \} \} \quad \gamma_{2c}(E) = 1 \]

\[ G_{1a}^{\uparrow IG} \]
\[ \eta_{2c} \triangleright G_{1a} = e_A \oplus e_B \text{ regular representation} \]

\[ \eta_{2c} \triangleright G_{1b} = e_A \oplus e_B \]

this means that

\[ A_{1a} \triangleright G \oplus B_{1a} \triangleright G \equiv A_{1b} \triangleright G \oplus B_{1b} \triangleright G \]

\[ \text{unocc} \rightarrow \Delta > t_1 \text{ occ} \]

\[ \text{occ} \rightarrow t_1 > \Delta \text{ unocc} \]

Let's generalize now to 2D (spinless)

\[ C_2 : (x, y) \rightarrow (-x, -y) \]
All these have the same irreps:

\[ \begin{array}{cc}
G & C_2 \\
A & 1 \\
0 & 1 \\
\end{array} \]

We want \( B^k(\{C_2, 10\}) \)
Eight possibilities

\[ A_{1a} = +1 \]
\[ A_{1b} = e^{-i k_x a} \]
\[ A_{1c} = e^{-i k_y b} \]
\[ A_{1d} = e^{-i(k_x a + k_y b)} \]

\[ B_{1a} = -1 \]
\[ B_{1b} = -e^{-i k_x a} \]
\[ B_{1c} = -e^{-i k_y b} \]
\[ B_{1d} = -e^{-i(k_x a + k_y b)} \]

In the 87

k_y
These are the eight possible configurations of $C_2$ expersnas compatible w/ having a single exponentially localized Wannier function.
All have an even # of negative eigenvalues

\[ h(k) = (\Delta + t_1 \cos k_x) \sigma_z + t_2 \sin k_x \sigma_y \]
\[ + \Delta \cos k_y \sigma_z + t_2 \sin k_y \sigma_x \]

\[ \sigma_z h(-k) \sigma_z = h(k) \quad C_2 \text{ symmetric} \]
$$h(k_x, k_y=0) = (2\Delta + t_1 \cos k_x) \sigma_z + t_2 \sin k_x \sigma_y$$

$$h(k_x, k_y=\pi) = t_1 \cos k_x \sigma_z + t_2 \sin k_x \sigma_y$$

If $2\Delta > t_1$, $h(k_x, k_y=0)$ is the Bloch Hamiltonian for a 1D chain in one of our two phases.

$h(k_x, k_y=\pi)$ is the other phase.
The occupied band's Wilson loop:

\[ W_{2\pi i \xi - \Omega} (k_x) \]
$W_{k_x R_y (y)}$ has center given by $R_y + \frac{\Psi(k_x)}{2\pi}$
$W_{k_x+2\pi} r_y(y) = W_{k_x} r_{y+1}(y)$

\( \Rightarrow \) we can FT to get localized Wannier functions

\( D \) - the # of times \( W_{2\pi y/\epsilon} \) winds from \(-\pi\) to \( \pi\)

\[
D = \frac{1}{2\pi i} \oint d(k) k [\Omega_\nu(k)] \quad \text{Clean } H
\]
Berry curd