

## Lecture 13

Today - More on the induced representation:

$G \supset G_k$  representation  $\rho_k$  of  $G_k$

$\eta = \rho_k \uparrow G$  is the rep. of  $G$  induced from  $\rho_k$

① What is the dimension of  $\eta$ ?  $\eta(g)$  is a matrix that acts on

$$W = \bigoplus_{i=1}^N V_i$$

$N$ : number of cosets of  $G_k$  - index  $[G:G_k]$

$$\dim W = (\dim V) [G:G_k]$$

$$\Rightarrow \dim \rho_k \uparrow G = (\dim \rho_k) [G:G_k]$$

$[G : G_k]$  is the # of vectors  
in the star at  $\vec{k}$ .

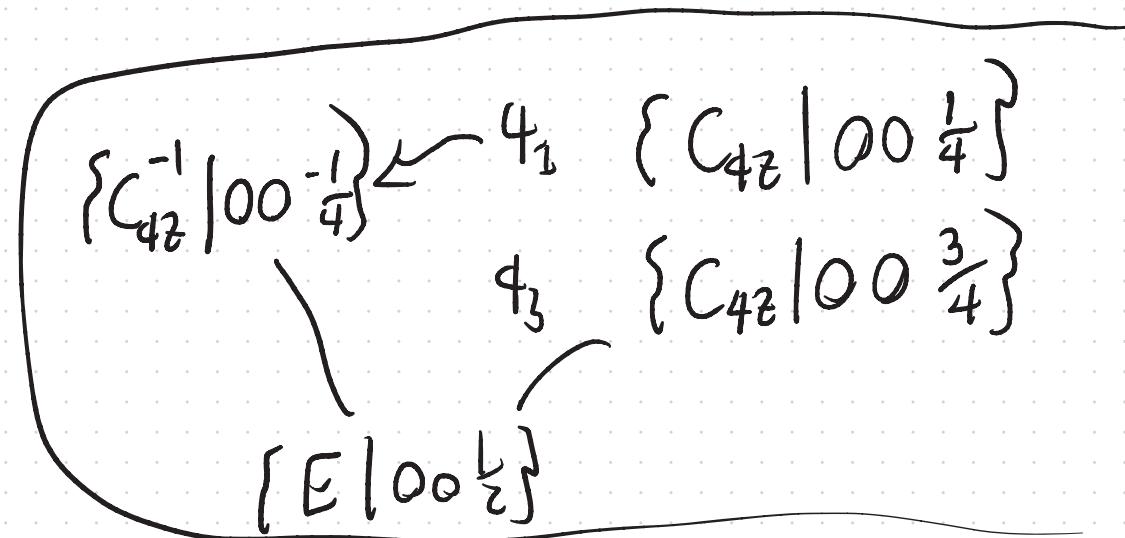
② What are the characters  $\chi_{\rho_k \cap G}$ ?

$$\chi_{\rho_k \cap G}(g) = \text{tr } \eta(g)$$

For each  $i$  we  
can ask is

$$g = g_i h g_i^{-1}$$

for some  $h \in G_k$



If yes, then the  $i$ -th block

$$\text{of } \eta(g) \text{ is } P_k(h) = P_k(g^{-1}gg_i)$$

else the  $i$ -th block is zero

$$\text{tr}(\eta(g)) = \sum_i \text{tr}[\text{ $i$ -th block of } \eta(g)]$$

$$\tilde{\chi}_{e_k}(g^{-1}gg_i) = \begin{cases} \chi_{e_k}(g^{-1}gg_i) & \text{if } g^{-1}gg_i \in G_k \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{i=1}^N \tilde{\chi}_{e_k}(g^{-1}gg_i)$$

③ If  $\rho_k$  and  $\rho'_k$  are isomorphic, then  
 $\rho_k^{\dagger}G \cong \rho'_k{}^{\dagger}G$

④ Frobenius Reciprocity

Let  $\rho_k$  be a rep of  $G_k \backslash G$ , and let  
 $\theta$  be a representation of  $G$

$$\langle \chi_{\rho_k^{\dagger}G}, \chi_{\theta} \rangle = \langle \chi_{\rho_k}, \chi_{\theta \downarrow G_k} \rangle$$

more generally

$\{$  homomorphisms from  $\mathbb{P}_k^1 PG$  to  $\Theta\}$

can be replaced  
w/ an integration  
for infinite  
grps

or imagine  
periodic  
b.c.s

$\simeq \{$  homomorphisms from  $\mathbb{P}_k$  to  $\Theta|_{PG}\}$

"induction is adjoint to restriction"

$$\langle \chi_{\mathbb{P}_k^1 PG}, \chi_\theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{P}_k^1 PG}(g^{-1}) \chi_\theta(g)$$

$$\langle \chi_{\mathbb{P}_k}, \chi_{\theta|_{PG}} \rangle = \frac{1}{|G_k|} \sum_{h \in G_k} \chi_{\mathbb{P}_k}(h^{-1}) \chi_\theta(h)$$

Proof sketch: first we want to show

$$\begin{aligned} & \left\{ \text{homomorphisms from } \mathbb{P}_k^1 G \text{ to } \Theta \right\} \\ & \simeq \left\{ \text{homomorphisms from } \mathbb{P}_k \text{ to } \Theta/G \right\} \end{aligned}$$

Say  $\mathbb{P}_k$  acts on some vector space  $V$

$\Theta$  acts on some vector space  $W$

given any map  $\tilde{\phi}: \bigoplus_{i=1}^N V_i \rightarrow W$  we want to  
construct a unique  $\phi: V \rightarrow W$

$$\theta(g)\tilde{\phi}(v) = \tilde{\phi}(e_k \gamma g(g)v) \text{ for } v \in \bigoplus_{i=1}^N V_i$$

$$\phi = \tilde{\phi}|_{V_1}$$

$$\theta(h)\phi(v) = \phi(e_k \gamma g(h)v) \text{ for all } v \in V_1$$

We also need to show how to construct  
 $\tilde{\phi}$  given  $\phi$

$$\phi(e_k(h)v) = \theta(h)\phi(v) \quad \text{for } h \in G_k$$

We want to extend to

$$\tilde{\phi} \text{ s.t. } \tilde{\phi}(e_k \cap G(g)v) = \theta(g)\tilde{\phi}(v)$$

$\forall v_i \in V_1$  we know that  $[e_k \cap G(g_i)]^+ v_i \in V_1$

$$\tilde{\phi}(v_i) = \phi([e_k \cap G(g_i)]^+ v_i)$$

$$E_k \cap G = \bigoplus_i \eta_i M_i$$

See Serre's book  
for details

$$\theta = \bigoplus_i \eta_i N_i$$

Ex

$$E_k \cap G = \underbrace{\eta_1 \oplus \eta_2 \oplus \eta_2}_{\hookrightarrow} \quad = 3 \stackrel{\oplus}{,} V_i$$

$$\theta = \underbrace{\eta_1 \oplus \eta_2 \oplus \eta_3}_{= W}$$

I want to parametrise maps from

$$\oplus V_j \rightarrow W$$

$$\begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & \lambda_1 \text{Id} & 0 & 0 \\ \eta_2 & 0 & \lambda_2 \text{Id} & 0 \\ \eta_3 & 0 & 0 & \lambda_3 \text{Id} \end{pmatrix}$$

What this tells us, is that states at all wavevectors in the  $\star$  of  $k$  that transform in the same space grp irrep are degenerate

Ex Mirror symmetry in 1d

$$G = \langle \{E|a\hat{x}\}, \{M_X|0\} \rangle$$

$$\vec{k} = 0; \quad M_X \vec{k} = 0 = \vec{k}$$

$$G_{\vec{r}} = G$$

$$\vec{k} = \frac{\pi}{a} \hat{x} \quad M_X \vec{k} = -\frac{\pi}{a} \hat{x} = \vec{k} - \frac{2\pi}{a} \hat{x}$$

$$\Rightarrow G_{\frac{\pi}{a} \hat{x}} = G_X = G$$

For these pts little grp irreps are space grp irreps, since the star has one arm

$$\vec{k} \neq 0, \vec{q} \vec{x} \Rightarrow m_x \vec{k} = -\vec{k} \neq \vec{k}$$

$$G_k = \langle \{E|\vec{a}\vec{x}\} \rangle$$

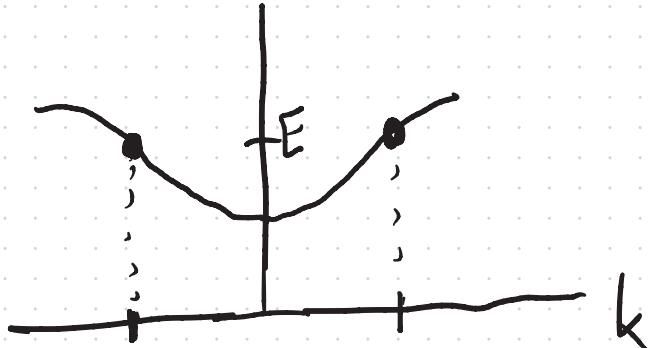
only one Irrep  $\rho_k(\vec{t}) = e^{-ik \cdot \vec{t}}$

lets construct  $\eta = \rho_k \tilde{\rho} G$

$$\eta(\vec{t}) = \begin{pmatrix} e^{-i\vec{k} \cdot \vec{t}} & 0 \\ 0 & e^{+i\vec{k} \cdot \vec{t}} \end{pmatrix}$$

$$e_k(M_X^{-1} \vec{t} M_X) = P_k(-\vec{t})$$

$$\gamma(M_X) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$E_{\pi k} = E_{\pi -k}$  by Schur's lemma, since they transform in the same irrep

This is consistent w/

$$B(g)^T H(k) B(g) = H(gk)$$

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What do we need to generalize to account for spin  $\frac{1}{2}$ ?

We constructed space/pt groups as  
Subgroups of  $O(3) \times \mathbb{R}^3$

But  $Spin-\frac{1}{2}$  particles transform in irreps of  
 $SU(2)$ , not  $\underline{SO(3)}$

$$e_{\frac{1}{2}}(R_{\theta, \hat{n}}) = e^{-i\theta \hat{n} \cdot (\vec{\sigma}_z)} \quad R_{\theta, \hat{n}}$$
$$R_{2\pi, \hat{n}} = E$$

$$\downarrow \\ \ell_2(R_{2\pi, \hat{n}}) = e^{-i\pi \hat{n} \cdot \vec{\sigma}} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \neq \ell_2(E)$$

We can construct a central extension of  $SU(3)$

by  $H = \{E, \bar{E}\}$   $\bar{E}$  is a  $2\pi$  rotation

$$\overset{\uparrow}{SU(2)}$$

Ex: let's consider  $222 = \{E, C_{2x}, C_{2y}, C_{2z}\}$

$$C_{2x}C_{2y} = C_{2y}C_{2x} = C_{2z}$$

$$P_{1/2}(C_{z\bar{z}}) = e^{-i\pi/2(\sigma_z)} = -i\sigma_z$$

$$P_{1/2}(C_{zx})P_{1/2}(C_{zy}) = (-i\sigma_x)(-i\sigma_y) = -i\sigma_z = P_{1/2}(C_{zz})$$

$$P_{1/2}(C_{zy})P_{1/2}(C_{zx}) = (-i\sigma_y)(-i\sigma_x) = +i\sigma_z$$

$$= -P_{1/2}(C_{zz})$$

$$e^{ic(C_{zx}, C_{zy})} = 1$$

$$e^{ic(C_{zy}, C_{zx})} = -1$$

$P_{1/2}$  is a projective rep

Alternatively, let  $222^d$

$$C_{2x} C_{2y} = \overline{C_{2z}}$$

$$C_{2x}^2 = C_{2y}^2 = C_{2z}^2 = \bar{E}$$

$$C_{2y} C_{2x} = \bar{E} C_{2z}$$

$$P_L(\bar{E}) = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$P_L$  is an ordinary rep. of  $222^d$

For  $Q(3)$ , there's a subtlety

for spatial inversion],  $I^2 =$

$$\begin{cases} E \leftarrow P_{in-} \\ \bar{E} \leftarrow P_{in+} \end{cases}$$

We use  $P_{in}$  b/c spatial inversion doesn't affect spin

$$P_{in} = SO(2) \times \{E, I\}$$

We can work w/ double groups and their representations

$$\eta(\bar{E}) = \begin{cases} +\text{Identity} - \text{coincide w/} \\ \quad \text{ordinary reps} \\ -\text{Identity} - \text{spin-}\frac{1}{2}\text{ reps} \end{cases}$$

f SOC vanishes

$$H = \bigoplus_i H_i \otimes O_0$$

completely redp  
of spin

$$\eta(\bar{E}) = \begin{cases} +\text{Identity} - \text{coincide w/ ordinary reps} & -\text{single-valued reps} \\ -\text{Identity} - \text{spin-}\frac{1}{2}\text{ reps} & -\text{double-valued reps} \end{cases}$$

$$\bar{E} = (E, \bar{E}) \in G \times H$$