PHYS 598 GTC Homework 3

1. Consider the projector product representation of the Wilson line,

$$W^{nm}_{k_\mu \leftarrow 0}(\mathbf{k}_\perp) = \left\langle u_{nk_\mu,\mathbf{k}_\perp} \right| \prod_{k'_\mu}^{k_\mu \leftarrow 0} P(k'_\mu, \mathbf{k}_\perp) \left| u_{m0,\mathbf{k}_\perp} \right\rangle , \tag{1}$$

   where

$$P(k_\mu, \mathbf{k}_\perp) = \sum_{n=1}^{N} \left| u_{nk_\mu,\mathbf{k}_\perp} \right\rangle \left\langle u_{nk_\mu,\mathbf{k}_\perp} \right| . \tag{2}$$

   Show that $W^{nm}_{k_\mu \leftarrow 0}(\mathbf{k}_\perp)$ satisfies

$$iD_\mu W = 0, \tag{3}$$

   where D_μ is the covariant derivative.
   Hint: start with the definition

$$\partial_\mu W^{nm}_{k_\mu \leftarrow 0}(\mathbf{k}_\perp) = \lim_{\Delta \to 0} \frac{1}{\Delta} \left(W^{nm}_{k_\mu + \Delta \leftarrow 0}(\mathbf{k}_\perp) - W^{nm}_{k_\mu \leftarrow 0}(\mathbf{k}_\perp) \right) \tag{4}$$

   of the derivative.

2. In this problem, we will show that

$$\det\left\{ W_{k_\mu \leftarrow 0}(\mathbf{k}_\perp) \right\} = \exp\left(i \int dk'_\mu \mathrm{tr}\left[A_\mu(k'_\mu, \mathbf{k}_\perp) \right] \right), \tag{5}$$

   where tr denotes the matrix trace. We will proceed via the following steps:

   (a) Show that Eq. (1) can be rewritten as

$$W_{k_\mu \leftarrow 0}(\mathbf{k}_\perp) = \lim_{\Delta \to 0} \prod_{k'_\mu}^{k_\mu \leftarrow 0} M(k'_\mu, \mathbf{k}_\perp, \Delta), \tag{6}$$

   where

$$M^{nm}(k'_\mu, \mathbf{k}_\perp, \Delta) = \left\langle u_{nk_\mu+\Delta,\mathbf{k}_\perp} \middle| u_{mk_\mu,\mathbf{k}_\perp} \right\rangle \tag{7}$$

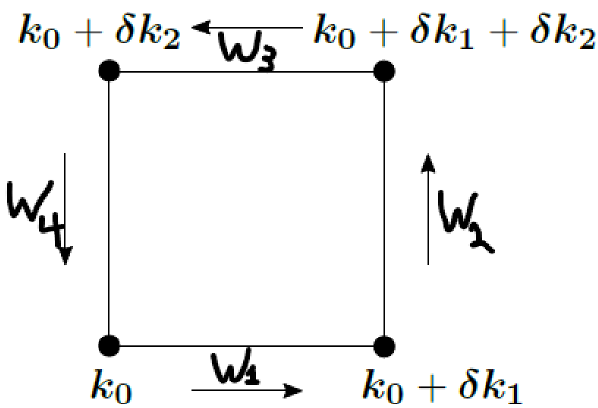
   (b) Show that $\det M(k'_\mu, \mathbf{k}_\perp, \Delta) = \exp\left(i\Delta \mathrm{tr}\left[A_\mu(k'_\mu, \mathbf{k}_\perp) \right] \right) + \mathcal{O}(\Delta^2)$.

   (c) Using (a) and (b), prove Eq. (5).

3. Consider the product of four Wilson lines around a small square of area $\delta k_1 \delta k_2$,

$$W = W_4 W_3 W_2 W_1 \tag{8}$$

depicted in the figure below.



(a) Using Stokes's theorem and the results of Problem 2, show that

$$\lim_{\delta k_1, \delta k_2 \to 0} \frac{1}{\delta k_1 \delta k_2} \mathrm{Im} \log \det W = \mathrm{tr} \Omega(\mathbf{k_0}), \tag{9}$$

where $\Omega(\mathbf{k}_0)$ is the Berry curvature.

(b) Show that the combination of inversion and time-reversal symmetry requires $\det W$ to be real

(c) Deduce the constraint on $\mathrm{tr}(\Omega(\mathbf{k_0}))$ implied by (a) and (b). What does this mean for Wannier functions when we have a single isolated band with inversion and time-reversal symmetry?

4. Consider the $\mathbf{k} \cdot \mathbf{p}$ Hamiltonian

$$H(\mathbf{k}) = v \mathbf{k} \cdot \vec{\sigma}, \tag{10}$$

where $v > 0$ and $\vec{\sigma}$ is a vector of three Pauli matrices.

(a) Find the energies $\epsilon_\pm(\mathbf{k})$ and the eigenstates $|u_{\pm,\mathbf{k}}\rangle$.

(b) Compute the Berry connection $A_{+\mu} = i \langle u_{+,\mathbf{k}} | \partial_\mu u_{+,\mathbf{k}} \rangle$ for the upper band.

(c) Find the Berry curvature $\Omega_+(\mathbf{k})$. What is the integral of $\Omega_+(\mathbf{k})$ over a constant energy surface? *Hint: Work in polar coordinates for as long as possible.*