PHYS 598 GTC Homework 2

1. One representation of a group that is particularly important is called the *regular representation*. If G is a finite group ($|G| < \infty$), then we can define the regular representation ρ as follows: We consider the $|G|$-dimensional complex vector space $\mathbb{C}^{|G|}$. There is one basis vector $\mathbf{e}_g$ per group element $g \in G$. We define the representation $\rho : G \to U(|G|)$ acting on this vector space as
$$\rho(g)\mathbf{e}_{g'} = \mathbf{e}_{gg'}$$
for basis vectors.

    (a) By looking at the matrix elements of $\rho(g)$, show that the character χ_ρ for the regular representation satisfies
    $$\chi_\rho(g) = \begin{cases} 0, & g \neq E_G \\ |G|, & g = E_G \end{cases},$$
    where E_G is the identity element in G.

    (b) Let $\{\eta_i\}$ be the set of irreps of G. Show that the regular representation is reducible and decomposes as
    $$\rho = \bigoplus_i [\dim(\eta_i)]\eta_i.$$

2. An important theorem about groups that we have used implicitly in proving Schur's lemma is known (confusingly) as the first isomorphism theorem. Here we will state and prove this theorem. Let G and H be two groups, and consider a function $\phi : G \to H$ from G to H. ϕ is called a *group homomorphism* if it is compatible with group multiplication in G and H, i.e. if
$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$$
for all $g_1, g_2 \in G$. Note that if H is the unitary group, then ϕ is a representation as we have defined it in class. For a group homomorphism ϕ, we can consider the following two sets:
$$\mathrm{Ker}(\phi) = \{g \in G \mid \phi(g) = E_H\},$$
$$\mathrm{Im}(\phi) = \{\phi(g) \mid g \in G\},$$
where E_H denotes the identity element in H.

    (a) Show that $\mathrm{Ker}(\phi)$ is a normal subgroup of G.

    (b) Show that $\mathrm{Im}(\phi)$ is a subgroup of H.

    (c) Using a coset decomposition, show that $\mathrm{Im}(\phi) \approx G/\mathrm{Ker}(\phi)$.

3. Consider the space group $C2/m$. We can take a presentation of this group with primitive Bravais lattice vectors

$$\mathbf{e}_1 = \frac{1}{2}(a\hat{\mathbf{x}} + b\hat{\mathbf{y}}),$$
$$\mathbf{e}_2 = \frac{1}{2}(a\hat{\mathbf{x}} - b\hat{\mathbf{y}})$$
$$\mathbf{e}_3 = c\hat{\mathbf{z}},$$

mirror symmetry m_y, and inversion symmetry about the origin.

(a) Write down a set of primitive reciprocal lattice vectors.

(b) Consider the following $\mathbf{k}$-vectors given in Cartesian components. What are their little groups?

i. $\Gamma = \vec{0}$

ii. $Y = \frac{2\pi}{b}\hat{\mathbf{y}}$

iii. $B = 2\pi\left(\frac{1}{3a}\hat{\mathbf{x}} + \frac{1}{7c}\hat{\mathbf{z}}\right)$

(c) Sketch the primitive unit cell for the reciprocal lattice (a 2D projection is fine), and identify the points that have m_y in their little group. How many independent m_y-invariant planes are there?

4. Consider the space group $P4_1$. By considering monodromy of little group representations, determine the smallest number of connected bands that is consistent with the compatibility relations. From this, what can you say about a crystal with $P4_1$ symmetry and two electrons per unit cell?