PHYS 598 GTC Homework 1

1. Let $G$ be a group, and let $H \subset G$ be a subgroup. Let’s get some practice with group theory basics by proving the following two statements:

   (a) Show that if the index of $H$ in $G$ is two (i.e. $|G : H| = 2$), then $H \triangleleft G$ is a normal subgroup of $G$.

   (b) Let $|G|$ and $|H|$ denote the number of elements in $G$ and $H$ respectively (called the order of the group). Using the coset decomposition of $G$, show that if $|G|$ is finite then $|G| = |G : H||H|$ (This result is known as Lagrange’s theorem).

2. Consider a group $\bar{G}$ isomorphic to the point group 6mm.

   (a) What is the order $|\bar{G}|$ of $\bar{G}$?

   (b) Show that there are three subgroups $H_1$, $H_2$, and $H_3$ of $\bar{G}$, all of which are isomorphic to 2mm.

   (c) Show that there exists $g \in \bar{G}$ such that $gH_1g^{-1} = H_2$, and $gH_2g^{-1} = H_3$. Are any of $H_1$, $H_2$, or $H_3$ normal?

3. Let’s get some practice with point groups in 2D. For the three decorated squares below, identify the point group that gives the symmetry of the decorated square:

   (a) 

   (b) 

   (c) 

4. Let’s get some practice reading space group symbols. Using the Bilbao Crystallographic Server, Bradley and Cracknell, or otherwise, answer the following questions about the space group $P222_1$ (# 17):
(a) What is the point group $\bar{G}$ of this space group?
(b) What is the Bravais lattice $T$ of this space group?
(c) Write a coset decomposition of $P\overline{2}2_1$ relative to $T$.

5. The most common Bravais lattice types (which you may have seen before) are primitive (P), body-centered (I), and face centered (F). Let’s look at these for a cubic system:

   (a) Write down a set of primitive Bravais lattice vectors for a primitive cubic lattice with side length $a$. What is the volume of the primitive unit cell?
   (b) Do the same for a body-centered cubic lattice (with the sides of the cube still of length $a$). What is the volume of the primitive body-centered unit cell?
   (c) Finally, do the same for a face-centered Bravais lattice. What is the volume of the primitive face-centered unit cell?