Lecture 4

Recap: Representation ρ of a group G

- a vector space V
- a group homomorphism $\rho : G \to U(V)$ to unitary operators on V

Let G be a group, $\rho : G \to U(V)$ a representation on a vector space V

$g \in G \quad \rho(g) \in U(V)$

Given a vector $|v\rangle \in V \quad \rho(g)|v\rangle \in V$

We can look for $W \subseteq V$ such that

$$\rho(g)|w\rangle \in W \text{ for all } g \in G,\ |w\rangle \in W$$

such a subspace is called an <u>invariant subspace</u>

given $W \subseteq V$, we can consider $W^{\perp}$ - orthogonal complement of W:

$$W^{\perp} = \{ |w_{\perp}\rangle \in V \mid \langle w_{\perp}|w\rangle = 0 \text{ for all } |w\rangle \in W \}$$

$$V = W \oplus W^{\perp}$$

every $|v\rangle \in V$ can be written uniquely as $|v\rangle = \overset{\in W}{|w\rangle} + \overset{\in W^{\perp}}{|w_{\perp}\rangle}$

Since $\rho(g)$ are all unitary, $W^\perp$ is <u>also</u> an invariant <u>subspace</u>

pf: let $|w\rangle \in W$, $|w^\perp\rangle \in W^\perp$

W invariant, $\rho(g)|w\rangle \in W$ for any $g \in G$

$$\Rightarrow \langle w^\perp|(\rho(g)|w\rangle) = 0$$

$$= (\langle w|(\rho(g))^\dagger|w^\perp\rangle)^* = 0$$

$$\Rightarrow \langle w|\rho(g^{-1})|w^\perp\rangle = 0$$

for any $g \in G$

for any $|w\rangle \in W$

for any $|w^\perp\rangle \in W^\perp$

$\Rightarrow \rho(g^{-1})|w^\perp\rangle \in W^\perp$

$\Rightarrow W^\perp$ is an invariant subspace

orthonormal basis for V: $\{|v_1\rangle, |v_2\rangle, |v_3\rangle, \ldots\}$

$\rho(g)$ as a matrix in this basis

$$\rho(g) = \begin{pmatrix} \langle v_1|\rho(g)|v_1\rangle & \langle v_1|\rho(g)|v_2\rangle & \cdots \\ \langle v_2|\rho(g)|v_1\rangle & \langle v_2|\rho(g)|v_2\rangle & \cdots \\ \vdots & & \\ \vdots & & \end{pmatrix}$$

$\rho: G \to U(V)$ is a homomorphism

$\rho(gg^{-1}) = \rho(g)\rho(g^{-1}) = E$

$= \rho(g)^\dagger \rho(g)\rho(g^{-1}) = \rho(g)^\dagger$

$\Rightarrow \rho(g^{-1}) = \rho(g)^\dagger$

But instead, we can pick a different basis

$\{|w_1\rangle, |w_2\rangle, |w_3\rangle \ldots\}$ be a basis for W
orthonormal

$\{|w_1^\perp\rangle, |w_2^\perp\rangle, |w_3^\perp\rangle \ldots\}$ be an orthonormal basis for $W^\perp$

a basis for V $\{|w_1\rangle, |w_2\rangle, \ldots, |w_1^\perp\rangle, |w_2^\perp\rangle \ldots\}$

$V = W \oplus W^\perp$

$$|v\rangle = \underbrace{\sum_i |w_i\rangle\langle w_i|v\rangle}_{\in W} + \underbrace{\sum_j |w_j^\perp\rangle\langle w_j|v\rangle}_{\in W^\perp}$$

in this basis

$$\rho(s) =: \begin{array}{cc} W & W^\perp \\ \left(\begin{array}{cc} \langle w_i|\rho(s)|w_j\rangle & \langle w_i|\rho(s)|w_j^\perp\rangle \\ \langle w_i^\perp|\rho(s)|w_j\rangle & \langle w_i^\perp|\rho(s)|w_j^\perp\rangle \end{array}\right) \end{array} \begin{array}{c} W \\ W^\perp \end{array} = \left(\begin{array}{c|c} \rho_W(s) & 0 \\ \hline 0 & \rho_{W^\perp}(s) \end{array}\right)$$

$= 0$

0

ρ_W and $\rho_{W^\perp}$ are themselves representations

up to a change of basis

$\rho \simeq \rho_W \oplus \rho_{W^\perp}$

"is unitarily equivalent to"

$\rho_1 \simeq \rho_2$ if $\exists U$ $U\rho_1(s)U^\dagger = \rho_2(s)$

$\rho_W, \rho_{W^\perp}$ are subrepresentations of ρ

ρ is a <u>reducible representation</u>

$\forall g$

Any representation that is not reducible is called <u>irreducible</u>

Note if ρ is irreducible then the only invariant subspaces of ρ are $\{\vec{0}\}$ and V

Trivial example: let G be any group, and $V = \mathbb{C}$

vector space of complex numbers

$$U(\mathbb{C}) = \{e^{i\phi} / \phi \in (0, 2\pi]\} = U(1)$$

$$\rho : g \to \rho(g) = 1 = e^{i0}$$

$$\rho(g_1 g_2) = 1 = \rho(g_1)\rho(g_2) = 1 \times 1$$

trivial representation - irreducible

Nontrivial example: $SU(2)$ we know: $V_{\frac{1}{2}} = \{|\uparrow\rangle, |\downarrow\rangle\}$

$$\rho_{\frac{1}{2}} : (\hat{n}, \theta) \to e^{-i\frac{\theta}{2}\vec{\sigma}\cdot\hat{n}}$$

On the vector space of two spin-$\frac{1}{2}$ particles:

$$V = V_{\frac{1}{2}} \otimes V_{\frac{1}{2}} = \{|\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle\}$$

$$\rho_{\frac{1}{2}\times\frac{1}{2}} : (\hat{n}, \theta) \to e^{-i\frac{\theta}{2}\vec{\sigma}_1\cdot\hat{n}} \otimes e^{-i\frac{\theta}{2}\vec{\sigma}_2\cdot\hat{n}}$$

Are there (nontrivial) invariant subspaces for $\rho_{\frac{1}{2}\times\frac{1}{2}}$?

$$\left\{ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \right\} = W$$ - singlet state

invariant subspace

$$W^{\perp} = \left\{ |\uparrow\uparrow\rangle, \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right), |\downarrow\downarrow\rangle \right\}$$ - invariant subspace: triplet space

in this basis $\rho_{\frac{1}{2}\times\frac{1}{2}}(\hat{n}, \theta) = \left(\begin{array}{c|c} 1 & \vec{0} \\ \hline \vec{0} & e^{-i\hat{n}\cdot\vec{L}\theta} \end{array} \right)$

spin-1 representation

$$\rho_{\frac{1}{2}\times\frac{1}{2}} \cong \rho_0 \oplus \rho_1 \quad \left(\text{“} \tfrac{1}{2} \otimes \tfrac{1}{2} = 0 \oplus 1 \text{”} \right)$$

Clebsch-Gordan coeffs: matrix elements of the unitary matrix that block-diagonalize ρ

Schur's Lemma ($2\frac{1}{2}$ parts)

Alternatively "Weyl unitarity trick" $\rho: G \to GL(V)$

$\langle v | w \rangle$ - Hermitian inner product

$$\langle\langle v | w \rangle\rangle = \sum_{g \in G} \langle \rho(g) v | \rho(g) w \rangle$$

Schur's Lemma part 1: let G be a group

$\rho_1 : G \to U(V_1)$

$\rho_2 : G \to U(V_2)$

two irreducible representations (irreps) of G

and a matrix $H : V_1 \to V_2$

$|v_1\rangle \in V_1$

$|v_2\rangle \in V_2$

$H|v_1\rangle \in V_2$

If $\underline{H\rho_1(g)} = \rho_2(g)H$ for all $g \in G$

then either: ① $H = 0$

② H is invertible

Pf: if $H = 0$ $\ker H = V_1$

$\operatorname{Im} H = \{\vec{0}\}$

if H is invertible $\qquad \ker H = \{\vec{0}\}$

$\operatorname{Im} H = V_2$

Lets look at $\ker H = \{|v\rangle \in V_1 \mid H|v\rangle = 0\}$

pick $|v_1\rangle \in \ker H$

$0 = \rho_2(g) H |v_1\rangle = H\big(\rho_1(g)|v_1\rangle\big)$

$\Rightarrow$ $|v_1\rangle \in \ker H$ then so is $\rho_1(g)|v_1\rangle$ for all $g \in G$

$\Rightarrow$ $\ker H$ is an invariant subspace of V_1

but ρ_1 is irreducible

$$\ker H = \begin{cases} \{\vec{0}\} & - \ H \text{ is one-to-one} \\ V_1 & - \ H = 0 \end{cases}$$

Now lets look at $\operatorname{Im} H = \{ |v_2\rangle \in V_2 \mid |v_2\rangle = H|v_1\rangle$ for some $|v_1\rangle \in V_1 \}$

$|w\rangle \in \operatorname{Im} H \ - \ |w\rangle = H|v\rangle \quad |v\rangle \in V_1$

$$\rho_2(g)|w\rangle = \rho_2(g) H|v\rangle = H\rho_1(g)|v\rangle$$

$$\underset{\operatorname{Im} H}{\cap} \quad \rightarrow \quad \operatorname{Im} H \text{ is an invariant subspace of } \rho_2$$

$$\operatorname{Im} H = \begin{cases} V_2 : & H \text{ is "onto"} \\ \vec{0} : & H = \vec{0} \end{cases}$$

Either $H = \vec{0}$ or

H is one-to-one and onto $\rightarrow$ H is invertible

<u>Part 2</u>: Suppose $V_1 = V_2 = V$ and $\rho_1 = \rho_2 = \rho$ are the same, ρ irreducible and finite-dimensional and assume we have H that satisfies part 1 assumption

$$H\rho(g) = \rho(g)H \Rightarrow [H, \rho(g)] = 0$$

then either: $H = 0$ or

$H = \lambda \, Id_V$ ← identity matrix

(if ρ is an irrep, the only matrix that commutes with every $\rho(g)$ is the identity)

Pf: Part 1 says $H = 0$ or H invertible

Assume H invertible → H a finite dim square matrix

⇒ H has at least one eigenvector $|v\rangle$ w/ eigenvalue λ

$\{|n\rangle \mid n = 0, 1, 2, \dots\}$

$a^\dagger$

$B = H - \lambda \, Id_V$ → $[B, \rho(g)] = 0$

⇒ $B = 0$ or B is invertible

$\uparrow$

$B|v\rangle = 0$ so B not invertible

$B = 0 \Rightarrow H = \lambda \, Id_V$

If G is a symmetry group H

$[H, \rho(g)] = 0$

$\{|\psi_i\rangle, \; i = 1, \dots N\}$ $\qquad \rho_{ji}(g)$

$$\rho(g)|\psi_i\rangle = \sum_{j=1}^{N} |\psi_j\rangle \langle\psi_j|\rho(g)|\psi_i\rangle$$

$$[H]_{ij} = \langle \psi_i | H | \psi_j \rangle$$

$$\sum_k [H]_{ik} \rho_{kj}(g) - \rho_{ik}(g)[H]_{kj} = 0$$

$$[H_{ij}] = E_i \delta_{ij} \rightarrow \boxed{\text{States transforming in an irrep of the symmetry group are degenerate}}$$

Ex: Nonrelativistic Hydrogen atom

Fixed n, $\{ |n l m_z\rangle \mid l = 0, \ldots n \quad m_z = -l, \ldots l \}$

$V_1 = \{|n\, l_1\, m_z\rangle\} \leftarrow$ spin l_1 irrep of $SO(3)$

$V_2 = \{|n\, l_2\, m_z'\rangle\} \leftarrow$ spin l_2 irrep $SO(3)$

$l_1 \neq l_2$ different dimensions

Part I: $\langle n\, l_1\, m_z | H | n\, l_2\, m_z' \rangle = \delta_{l_1 l_2}$

Part 2: $\langle n\, l_1\, m_z | H | n\, l_1\, m_z' \rangle = E\, \delta_{m_z m_z'}$

Part 2.5.

let G be a group

$\rho_1 : G \to U(V_1)$

$\rho_2 : G \to U(V_2)$ finite dim irreps

$$\rho_2(g) H = H \rho_1(g)$$

H invertible $\to$ $\rho_1 \cong \rho_2$

rep of $SU(2)$

$\ell \in \mathbb{Z}$ representation of $SO(3)$

$\ell \in \mathbb{Z} + \frac{1}{2}$

2x2 unitary matrices w/ determinant 1

$$e^{-i\frac{\theta}{2}\vec{\sigma}\cdot\hat{n}}$$